Mathematical Economics
Final Exam

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Name: ____

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 180 minutes.
- This exam has 7 questions on 9 pages excluding the cover page, for a total of 110 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- You can detach the last empty page and use it as a scratch sheet.
- Use the notation x_{il} for agent i's consumption of good l.
- Unless otherwise stated, normalize the price of goods so that $p_1 = 1$.

Question:	1	2	3	4	5	6	7	Total
Points:	18	8	20	10	14	20	20	110
Score:								

1. (a) (2 points) Legibly write down your name in the designated area on the cover and carefully read the instructions.

   (b) (4 points) What is the statement of the separating hyperplane theorem? (There are weak and strong versions. Either of them is fine.)

   (c) (4 points) Let R_m be the market return and R_j be the return of an asset. What is the definition of the beta of this asset?

   (d) (4 points) Suppose that you are an investment advisor. You know that the average risk tolerance in the economy is $\bar{\tau} = 0.5$ and your client has risk tolerance $\tau_i = 0.8$. Your client has 1 million dollars to invest. Assuming that the capital asset pricing model (CAPM) is true, what is your investment advice?

   (e) (4 points) Suppose that the expected market return is 7% and the risk-free rate is 1%. If an asset has $\beta = 1.5$, what is its expected return?

2. (8 points) Suppose that there are two goods. An agent has endowment $(e_1, e_2) = (5, 3)$ and utility function

$$u(x_1, x_2) = \log(x_1 - 2) + 2\log(x_2 - 1).$$

Compute the demand of the agent.

3. Consider an economy with two countries, $i = A, B$, and three consumption goods, $l = 1, 2, 3$. Both countries have labor endowment $e_A = e_B = 3$. The utility functions are

$$u_A(x_1, x_2, x_3) = \frac{1}{2}\log x_1 + \frac{1}{4}\log x_2 + \frac{1}{4}\log x_3,$$
$$u_B(x_1, x_2, x_3) = \frac{1}{3}\log x_1 + \frac{1}{3}\log x_2 + \frac{1}{3}\log x_3.$$

Each country can produce the consumption goods from labor using the linear technology $y = a_{il}e$, where e is labor input, y is output of good l, and $a_{il} > 0$ is the productivity. Assume that productivities are

$$(a_{A1}, a_{A2}, a_{A3}) = (4, 2, 2),$$
$$(a_{B1}, a_{B2}, a_{B3}) = (1, 1, 2).$$

(a) (4 points) What is the definition of comparative advantage of country A over B? Compute the comparative advantage for each industry.

(b) (4 points) Given the price $p = (p_1, p_2, p_3)$ and the wage w_A of country A, compute the demand of country A.

(c) (4 points) Assuming that both countries produce good 2 in free trade and setting $p_2 = 1$, compute p_1, p_3, w_A, w_B.

(d) (4 points) Compute the free trade equilibrium consumption in each country.

(e) (4 points) Compute the labor allocation across each industry for each country.

4. Consider a world with L goods indexed by $l = 1, \ldots, L$. Let $p = (p_1, \ldots, p_L)$ be the vector of world prices. Suppose that a small country (hence it does not affect world price p) has I citizens indexed by $i = 1, \ldots, I$, and let $u_i(x)$ be the (locally nonsatiated) utility function of agent i and e_i be the initial endowment.

   (a) (5 points) Suppose that the government is adopting some trade policy (tariff, quotas, etc.), and the equilibrium allocation is (x_i). If the government is neither running a trade surplus nor a deficit, show that it must be

$$\sum_{i=1}^{I} p \cdot (e_i - x_i) = 0.$$

   (b) (5 points) Find a trade policy that weakly Pareto improves the initial trade policy. Explain why your policy is weakly Pareto improving.

5. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility functions are

$$\begin{aligned} u_1(x_1, x_2) &= x_1^2 x_2, \\ u_2(x_1, x_2) &= x_1 x_2^2, \\ u_3(x_1, x_2) &= x_1 x_2. \end{aligned}$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (22, 8)$.

(a) (3 points) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

(b) (3 points) Compute the free trade equilibrium price and allocation.

(c) (3 points) Compute the utility level of each agent and determine who gained from trade and who lost.

(d) (5 points) Find a tax scheme in country A such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

6. Time is denoted by $t = 0, 1$. Suppose that the risk-free rate is $1 + r$. The stock price of a firm at $t = 0$ is $q_0 = 1$. There are two states at $t = 1$, up (u) and down (d). The stock price in each state at $t = 1$ is

$$q_1 = \begin{cases} 1 + \mu + \sigma, & (s = u) \\ 1 + \mu - \sigma, & (s = d) \end{cases}$$

where $\mu < \sigma < 1 + \mu$.

(a) (5 points) Let π_s be the state price of state s. Write down two equations that π_u, π_d satisfy.

(b) (5 points) Compute the state prices.

(c) (7 points) Suppose that the firm issues a *convertible bond* that promises to pay 1 in each state. By definition, by issuing the convertible bond, the firm has the option of either paying the promised payoff or converting the bond to a stock and delivering the stock. Compute the $t = 0$ price of the convertible bond.

(d) (3 points) Is the interest rate on the convertible bond higher or lower than the risk-free rate?

7. Time is denoted by $t = 0, 1$. There are J assets indexed by $j = 1, \dots, J$. At $t = 1$, there are S states indexed by $s = 1, \dots, S$. Asset j trades at price q_j at $t = 0$ and pays A_{sj} in state s at $t = 1$.

   (a) (5 points) What is the definition of an arbitrage?

(b) (5 points) What is the statement of the fundamental theorem of asset pricing?

(c) (10 points) (extra credit) Prove the fundamental theorem of asset pricing.

Use as scratch sheet.

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