Mathematical Economics
Final Exam

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Name: ____

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 180 minutes.
- This exam has 7 questions on 10 pages excluding the cover page, for a total of 110 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- You can detach the last empty page and use it as a scratch sheet.
- Use the notation x_{il} for agent i's consumption of good l.
- Unless otherwise stated, normalize the price of goods so that $p_1 = 1$.

Question:	1	2	3	4	5	6	7	Total
Points:	18	8	20	10	14	20	20	110
Score:								

1. (a) (2 points) Legibly write down your name in the designated area on the cover and carefully read the instructions.

   (b) (4 points) What is the statement of the separating hyperplane theorem? (There are weak and strong versions. Either of them is fine.)

   Solution: If sets C, D are nonempty, convex, and $C \cap D = \emptyset$, then there exists a hyperplane that separates them. More precisely, there exists a nonzero vector a such that

   $$a \cdot x \leq a \cdot y$$

   for all $x \in C$ and $y \in D$. (The strong version says that if C, D are nonempty, convex, and C is closed and D is compact, then the above inequality is strict.)

   (c) (4 points) Let R_m be the market return and R_j be the return of an asset. What is the definition of the beta of this asset?

   Solution: $\beta_j = \frac{\text{Cov}[R_m, R_j]}{\text{Var}[R_m]}$.

   (d) (4 points) Suppose that you are an investment advisor. You know that the average risk tolerance in the economy is $\bar{\tau} = 0.5$ and your client has risk tolerance $\tau_i = 0.8$. Your client has 1 million dollars to invest. Assuming that the capital asset pricing model (CAPM) is true, what is your investment advice?

   Solution: The client should invest fraction of wealth $\tau_i/\bar{\tau} = 1.6$ in the market portfolio, and $1 - \tau_i/\bar{\tau} = -0.6$ in the risk-free asset. Therefore the client should borrow 0.6 million and invest 1.6 million in the stock market. (1 point for mentioning the market portfolio, 1 point for mentioning the risk-free asset, and 2 points for the correct numbers.)

   (e) (4 points) Suppose that the expected market return is 7% and the risk-free rate is 1%. If an asset has $\beta = 1.5$, what is its expected return?

   Solution: By the covariance pricing formula,

   $$\text{E}[R_j] - R_f = \beta_j(\text{E}[R_m] - R_f).$$

   Therefore the expected return is $1.01 + 1.5 \times (1.07 - 1.01) = 1.1$, so 10%. (2 points for the correct answer, 2 points for derivation.)

2. (8 points) Suppose that there are two goods. An agent has endowment $(e_1, e_2) = (5, 3)$ and utility function

   $$u(x_1, x_2) = \log(x_1 - 2) + 2\log(x_2 - 1).$$

   Compute the demand of the agent.

Solution: The agent needs to consume 2 of good 1 and 1 of good 2 for subsistence. Then essentially the agent has (nonsubsistence) endowment $(5,3)-(2,1)=(3,2)$ and Cobb-Douglas utility function, so the demand is

$$\begin{aligned}(x_1,x_2) &= \left(2+\frac{1}{3}(3+2p), 1+\frac{2}{3p}(3+2p)\right)\\ &= \left(3+\frac{2}{3}p, \frac{7}{3}+\frac{2}{p}\right).\end{aligned}$$

Alternatively, you can set up the Lagrangian and solve in the usual way.

3. Consider an economy with two countries, $i=A,B$, and three consumption goods, $l=1,2,3$. Both countries have labor endowment $e_A=e_B=3$. The utility functions are

$$\begin{aligned}u_A(x_1,x_2,x_3) &= \frac{1}{2}\log x_1+\frac{1}{4}\log x_2+\frac{1}{4}\log x_3,\\ u_B(x_1,x_2,x_3) &= \frac{1}{3}\log x_1+\frac{1}{3}\log x_2+\frac{1}{3}\log x_3.\end{aligned}$$

Each country can produce the consumption goods from labor using the linear technology $y=a_{il}e$, where e is labor input, y is output of good l, and $a_{il}>0$ is the productivity. Assume that productivities are

$$\begin{aligned}(a_{A1},a_{A2},a_{A3}) &= (4,2,2),\\ (a_{B1},a_{B2},a_{B3}) &= (1,1,2).\end{aligned}$$

(a) (4 points) What is the definition of comparative advantage of country A over B? Compute the comparative advantage for each industry.

Solution: The comparative advantage is the ratio of productivities a_{Al}/a_{Bl} (2 points). Therefore they are

$$\left(\frac{a_{A1}}{a_{B1}}, \frac{a_{A2}}{a_{B2}}, \frac{a_{A3}}{a_{B3}}\right) = (4,2,1).$$

(2 points for numerical values.)

(b) (4 points) Given the price $p=(p_1,p_2,p_3)$ and the wage w_A of country A, compute the demand of country A.

Solution: The income of country A is $w_A\times e_A=3w_A$. Since the utility function is Cobb-Douglas, the demand is

$$(x_{A1},x_{A2},x_{A3}) = \left(\frac{3w_A}{2p_1}, \frac{3w_A}{4p_2}, \frac{3w_A}{4p_3}\right).$$

(c) (4 points) Assuming that both countries produce good 2 in free trade and setting $p_2 = 1$, compute p_1, p_3, w_A, w_B.

Solution: If country i produces good l, by profit maximization (zero profit) it must be $p_l a_{il} = w_i$. Setting $p_2 = 1$, we get $w_A = a_{A2} = 2$, $w_B = a_{B2} = 1$, $p_1 = w_A/a_{A1} = 2/4 = 1/2$, and $p_3 = w_B/a_{B3} = 1/2$. (1 point for each of p_1, p_3, w_A, w_B.)

(d) (4 points) Compute the free trade equilibrium consumption in each country.

Solution: Substituting the prices into the above formula, we get

$$(x_{A1}, x_{A2}, x_{A3}) = \left(\frac{3 \times 2}{3 \times 2(1/2)}, \frac{2}{4}, \frac{3 \times 2}{4(1/2)} \right) = (6, 3/2, 3).$$

Similarly, the consumption of country B is

$$(x_{B1}, x_{B2}, x_{B3}) = \left(\frac{3}{3(1/2)}, \frac{3}{3}, \frac{3}{3(1/2)} \right) = (2, 1, 2).$$

(e) (4 points) Compute the labor allocation across each industry for each country.

Solution: In order to produce

$$x_{A1} + x_{B1} = 6 + 2 = 8$$

units of good 1, country A has to hire labor

$$e_{A1} = 8/4 = 2.$$

Therefore $e_{A2} = 1$ and $e_{A3} = 0$.
In order to produce

$$x_{A3} + x_{B3} = 3 + 2 = 5$$

units of good 3, country B has to hire labor

$$e_{B3} = 5/2 = \frac{5}{2}.$$

Therefore $e_{B2} = \frac{1}{2}$ and $e_{B1} = 0$.
Then the world output of good 2 is

$$2 \times 1 + 1 \times \frac{1}{2} = \frac{5}{2} = x_{A2} + x_{B2},$$

so the markets clear.

4. Consider a world with L goods indexed by $l = 1, \dots, L$. Let $p = (p_1, \dots, p_L)$ be the vector of world prices. Suppose that a small country (hence it does not affect world price p) has I citizens indexed by $i = 1, \dots, I$, and let $u_i(x)$ be the (locally nonsatiated) utility function of agent i and e_i be the initial endowment.

   (a) (5 points) Suppose that the government is adopting some trade policy (tariff, quotas, etc.), and the equilibrium allocation is (x_i). If the government is neither running a trade surplus nor a deficit, show that it must be

$$\sum_{i=1}^{I} p \cdot (e_i - x_i) = 0.$$

Solution: The value of domestic consumption and endowment (GDP) evaluated at the world price are $\sum_{i=1}^{I} p \cdot x_i$ and $\sum_{i=1}^{I} p \cdot e_i$, respectively. Since the government runs a balanced budget, they must be equal. Therefore

$$\sum_{i=1}^{I} p \cdot x_i = \sum_{i=1}^{I} p \cdot e_i \iff \sum_{i=1}^{I} p \cdot (e_i - x_i) = 0.$$

   (b) (5 points) Find a trade policy that weakly Pareto improves the initial trade policy. Explain why your policy is weakly Pareto improving.

Solution: Make transfers such that the initial equilibrium allocation is just affordable under free trade, so let

$$p \cdot x_i = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i).$$

By the previous question, we have $\sum_{i=1}^{I} t_i = 0$, so the government budget balances. Since x_i is affordable, the resulting free trade allocation (x_i^f) weakly Pareto dominates the initial equilibrium allocation (x_i).

5. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility functions are

$$\begin{aligned} u_1(x_1, x_2) &= x_1^2 x_2, \\ u_2(x_1, x_2) &= x_1 x_2^2, \\ u_3(x_1, x_2) &= x_1 x_2. \end{aligned}$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (22, 8)$.

   (a) (3 points) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

Solution: Using the Cobb-Douglas formula, the demand of agents 1, 2 are

$$(x_{11}, x_{12}) = \left(\frac{2(3+3p)}{3}, \frac{3+3p}{3p}\right) = (2+2p, 1+1/p),$$
$$(x_{21}, x_{22}) = \left(\frac{3+3p}{3}, \frac{2(3+3p)}{3p}\right) = (1+p, 2+2/p).$$

Therefore by market clearing we have

$$(2+2p) + (1+p) = 3+3 \iff p = 1.$$

The demand is $(x_{11}, x_{12}) = (4, 2)$, $(x_{21}, x_{22}) = (2, 4)$. The utility level is $u_1^a = 4^2 \cdot 2 = 32$ and $u_2^a = 2 \cdot 4^2 = 32$.

(b) (3 points) Compute the free trade equilibrium price and allocation.

Solution: Using the Cobb-Douglas formula, the demand of agent 3 is

$$(x_{31}, x_{32}) = \left(\frac{22+8p}{2}, \frac{22+8p}{2p}\right) = (11+4p, 4+11/p).$$

Hence by market clearing, we have

$$(2+2p) + (1+p) + (11+4p) = 3+3+22 \iff p = 2.$$

Again using the Cobb-Douglas formula, the consumption of each agents are $(x_{11}, x_{12}) = (6, 3/2)$, $(x_{21}, x_{22}) = (3, 3)$, and $(x_{31}, x_{32}) = (19, 19/2)$.

(c) (3 points) Compute the utility level of each agent and determine who gained from trade and who lost.

Solution: Utility levels are

$$u_1^f = 6^2 \cdot 3/2 = 54 > 32 = u_1^a,$$
$$u_2^f = 3 \cdot 3^2 = 27 < 32 = u_2^a,$$
$$u_3^f = 19 \cdot 19/2 = \frac{361}{2} > 176 = 22 \cdot 8 = u_3^a.$$

Therefore agents 1 and 3 gained from trade and agent 2 lost.

(d) (5 points) Find a tax scheme in country A such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

Solution: Consider a hypothetical economy in which agents start with the autarky allocation $e_1' = (4, 2)$, $e_2' = (2, 4)$, and $e_3' = (22, 8)$. Using the

Cobb-Douglas formula, the demand is

$$(x_{11}, x_{12}) = \left(\frac{2(4+2p)}{3}, \frac{4+2p}{3p}\right),$$
$$(x_{21}, x_{22}) = \left(\frac{2+4p}{3}, \frac{2(2+4p)}{3p}\right),$$
$$(x_{31}, x_{32}) = \left(\frac{22+8p}{2}, \frac{22+8p}{2p}\right).$$

The market clearing condition is

$$\frac{2(4+2p)}{3} + \frac{2+4p}{3} + \frac{22+8p}{2} = 4+2+22 \iff p = \frac{41}{20}.$$

This price should be the free trade price after the tax and transfer. To make the autarky equilibrium allocation just affordable, taxes should be set such that $p \cdot x_i^a = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i^a)$. Therefore

$$t_1 = (1, 41/20) \cdot (3-4, 3-2) = -1 + \frac{41}{20} = \frac{21}{20},$$
$$t_2 = (1, 41/20) \cdot (3-2, 3-4) = 1 - \frac{41}{20} = -\frac{21}{20}.$$

With these tax/transfers and price, since the autarky allocation is affordable, the equilibrium allocation Pareto dominates the autarky allocation.

6. Time is denoted by $t = 0, 1$. Suppose that the risk-free rate is $1 + r$. The stock price of a firm at $t = 0$ is $q_0 = 1$. There are two states at $t = 1$, up (u) and down (d). The stock price in each state at $t = 1$ is

$$q_1 = \begin{cases} 1 + \mu + \sigma, & (s = u) \\ 1 + \mu - \sigma, & (s = d) \end{cases}$$

where $\mu < \sigma < 1 + \mu$.

(a) (5 points) Let π_s be the state price of state s. Write down two equations that π_u, π_d satisfy.

Solution:

$$1 = (1+r)\pi_u + (1+r)\pi_d,$$
$$1 = (1+\mu+\sigma)\pi_u + (1+\mu-\sigma)\pi_s.$$

(b) (5 points) Compute the state prices.

Solution:

$$
\begin{aligned}
\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1+r & 1+r \\ 1+\mu+\sigma & 1+\mu-\sigma \end{bmatrix} \begin{bmatrix} \pi_u \\ \pi_d \end{bmatrix} \\
\iff \begin{bmatrix} \pi_u \\ \pi_d \end{bmatrix} &= \frac{1}{2(1+r)(-\sigma)} \begin{bmatrix} 1+\mu-\sigma & -(1+r) \\ -(1+\mu+\sigma) & 1+r \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{2(1+r)} \begin{bmatrix} 1-(\mu-r)/\sigma \\ 1+(\mu-r)/\sigma \end{bmatrix}.
\end{aligned}
$$

(c) (7 points) Suppose that the firm issues a *convertible bond* that promises to pay 1 in each state. By definition, by issuing the convertible bond, the firm has the option of either paying the promised payoff or converting the bond to a stock and delivering the stock. Compute the $t = 0$ price of the convertible bond.

Solution: Since $\mu < \sigma$, the stock price in the down state $1 + \mu - \sigma$ is less than the promised payoff, 1. Therefore the firm will convert the bond to a stock only in the down state. Therefore the price of the convertible bond is

$$
\begin{aligned}
q_b &= \pi_u 1 + \pi_d(1+\mu-\sigma) = (\pi_u + \pi_d) + (\mu - \sigma)\pi_d \\
&= \frac{1}{1+r}\left[1 + \frac{1}{2}(\mu-\sigma)(1+(\mu-r)/\sigma)\right].
\end{aligned}
$$

(d) (3 points) Is the interest rate on the convertible bond higher or lower than the risk-free rate?

Solution: The interest rate of the convertible bond is $1 + r_b = 1/q_b$. Since $\mu - \sigma < 0$, we can see that

$$\frac{1}{1+r_b} = q_b < \frac{1}{1+r} \iff r_b > r,$$

so the interest rate of the convertible bond is higher than the risk free-rate. (This question can be answered without any calculation: since the convertible bond is just a bond with an option of converting by the issuer (not the investor), conversion can only harm the investor, and therefore the interest rate should be higher.)

7. Time is denoted by $t = 0, 1$. There are J assets indexed by $j = 1, \dots, J$. At $t = 1$, there are S states indexed by $s = 1, \dots, S$. Asset j trades at price q_j at $t = 0$ and pays A_{sj} in state s at $t = 1$.

   (a) (5 points) What is the definition of an arbitrage?

Solution: An arbitrage is something that costs nothing and pays something (*i.e.*, a free lunch). More precisely, a portfolio $\theta = (\theta_1, \dots, \theta_J)$ is an arbitrage if $\sum_{j=1}^{J} q_j\theta_j \leq 0$ and $\sum_{j=1}^{J} A_{sj}\theta_j \geq 0$ for all s, with at least one strict inequality.

(b) (5 points) What is the statement of the fundamental theorem of asset pricing?

Solution: There is no arbitrage if and only if there exists state prices $\pi_s > 0$ such that $q_j = \sum_{s=1}^{S} \pi_s A_{sj}$.

(c) (10 points) (extra credit) Prove the fundamental theorem of asset pricing.

Solution: See the lecture note.

Use as scratch sheet.

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