Mathematical Economics
Midterm Exam 1

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Name: ______________________________

Instructions:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 80 minutes.
- This exam has 6 questions on 5 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Submit your entire exam sheet before leaving the room, even if some parts are empty or you intend to drop the class.

Question:	1	2	3	4	5	6	Total
Points:	20	20	10	10	20	20	100
Score:							

1. (a) (5 points) Suppose that there are 2 goods, an agent has endowment $e = (e_1, e_2)$, and the price vector is $p = (p_1, p_2)$. Compute the wealth of the agent.

Solution: $w = p \cdot e = p_1 e_1 + p_2 e_2$.

(b) (5 points) Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$, the 2-dimensional Euclidean space. Give the exact definition for each of the following vector inequalities: $y \geq x$, $y > x$, and $y \gg x$.

Solution: If there are L goods labeled by $l = 1, 2, \ldots, L$, then

- $y \geq x$ if and only if $y_l \geq x_l$ for all l (2 points),
- $y > x$ if and only if $y_l \geq x_l$ for all l and $y_l > x_l$ for some l (2 points),
- $y \gg x$ if and only if $y_l > x_l$ for all l (1 points).

(c) (5 points) What is the definition of a utility function being weakly monotonic?

Solution: $u(x)$ is weakly monotonic if for all $y \geq x$ we have $u(y) \geq u(x)$ (2 points), and $u(y) > u(x)$ if $y \gg x$ (3 points).

(d) (5 points) Consider the sets

$$C = \left\{(x, y) \in \mathbb{R}^2_+ \,\middle|\, xy \geq 6\right\},$$
$$D = \left\{(x, y) \in \mathbb{R}^2_+ \,\middle|\, x < 3, y < 2\right\}.$$

Can C, D be separated? Answer yes or no. If yes, provide the equation of a separating hyperplane. If no, explain why.

Solution: Yes (2 points). The boundary of C is $y = 6/x$, a hyperbola. Since the derivative is $\mathrm{d}y/\mathrm{d}x = -6x^{-2}$, the tangent of C at the point $(x, y) = (3, 2)$ is $y = -\frac{2}{3}(x - 3) + 2 = -\frac{2}{3}x + 4$.

2. Consider an agent with utility function

$$u(x_1, x_2) = 2\sqrt{x_1} + 4\sqrt{x_2}.$$

Assume that the price is (p_1, p_2) and the agent has wealth w.

(a) (5 points) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$.

Solution:

$$L(x_1, x_2, \lambda) = 2\sqrt{x_1} + 4\sqrt{x_2} + \lambda(w - p_1 x_1 - p x_2).$$

(b) (5 points) Derive the first-order condition with respect to x_1 and x_2.

Solution:

$$x_1: \quad \frac{1}{\sqrt{x_1}} - \lambda p_1 = 0,$$
$$x_2: \quad \frac{2}{\sqrt{x_2}} - \lambda p_2 = 0.$$

(c) (5 points) Express the demand x_1, x_2 in terms of λ, p_1, p_2.

Solution: Solving above, we get $x_1 = (\lambda p_1)^{-2}$ and $x_2 = 4(\lambda p_2)^{-2}$.

(d) (5 points) Solve for the demand (x_1, x_2) using only w, p_1, p_2.

Solution: By the budget constraint, we get

$$w = p_1 x_1 + p_2 x_2 = \lambda^{-2}(1/p_1 + 4/p_2) \iff \lambda^{-2} = \frac{p_1 p_2 w}{4p_1 + p_2}.$$

Substituting in the previous result, we get

$$(x_1, x_2) = \left(\frac{p_2}{p_1} \frac{w}{4p_1 + p_2}, \frac{p_1}{p_2} \frac{4w}{4p_1 + p_2} \right).$$

3. Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy and $\{p, (x_i)\}$ be an equilibrium, where $p = (p_1, \ldots, p_L)$ is the price vector.

(a) (5 points) Assume $p_l < 0$. If an agent buys $\epsilon > 0$ of extra good l, how much more (or less) will his or her wealth be?

Solution: By buying ϵ of extra good l, the agent pays $p_l \epsilon < 0$. Therefore his or her wealth will increase by $-p_l \epsilon > 0$.

(b) (5 points) Prove that if we assume free disposal and at least one agent has a locally nonsatiated utility function, then prices cannot be negative.

Solution: Suppose that agent i has a locally nonsatiated utility function and $p_l < 0$. Consider his demand x_i. If he buys $\epsilon > 0$ of extra good l and throws them away, his position will still be x_i but he will have $-p_l \epsilon > 0$ of extra wealth. Then he can spend it to increase his utility, so x_i cannot be optimal in his budget constraint, which is a contradiction. Therefore $p_l \geq 0$.

4. Consider an economy with three goods labeled by $l = 1, 2, 3$. Let p_l be the price of good l. Without loss of generality, set $p_1 = 1$. Suppose an agent has wealth w and utility function

$$u(x_1, x_2, x_3) = x_1 - \frac{1}{2x_2^2} - x_3 \log x_3 + x_3.$$

(a) (3 points) What is the name of this type of utility function?

Solution: Quasi-linear.

(b) (7 points) Compute the demand of the agent. You may assume that good 1 can be consumed in positive or negative amounts.

Solution: The problem is to maximize $u(x)$ subject to $x_1+p_2x_2+p_3x_3 \leq w$. Since $u(x)$ is strongly monotonic (*i.e.*, increasing in every argument), the budget constraint binds: $x_1+p_2x_2+p_3x_3 = w$. Solving for x_1 and substituting into the utility function, it suffices to maximize

$$(w - p_2x_2 - p_3x_3) - \frac{1}{2x_2^2} - x_3 \log x_3 + x_3.$$

Therefore

$$0 = \frac{1}{x_2^3} - p_2 \qquad \iff \quad x_2 = p_2^{-\frac{1}{3}},$$
$$0 = -\log x_3 - p_3 \qquad \iff \quad x_3 = \mathrm{e}^{-p_3}.$$

Using the budget constraint, we get

$$x_1 = w - p_2x_2 - p_3x_3 = w - p_2^{\frac{2}{3}} - p_3\mathrm{e}^{-p_3}.$$

5. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility function of each agent is

$$u_A(x_1, x_2) = x_1 + 2\sqrt{x_2},$$
$$u_B(x_1, x_2) = x_1 + 2\log x_2.$$

Suppose that the initial endowments are $(e_1^A, e_2^A) = (4, 5)$ and $(e_1^B, e_2^B) = (10, 3)$.

(a) (10 points) Consider the problem of maximizing the sum of utilities,

$$\begin{aligned} &\text{maximize} && u_A(x_1^A, x_2^A) + u_B(x_1^B, x_2^B) \\ &\text{subject to} && x_1^A + x_1^B \leq e_1^A + e_1^B, \\ & && x_2^A + x_2^B \leq e_2^A + e_2^B. \end{aligned}$$

Let $(x_1^A, x_2^A, x_1^B, x_2^B)$ be a solution. Compute x_2^A and x_2^B.

Solution: Since the utility functions are quasi-linear, it suffices to maximize the sum of the nonlinear part. For notational simplicity, let $x_2^A = y$ and $x_2^B = z$. Then the problem becomes

$$\begin{aligned} &\text{maximize} && 2\sqrt{y} + 2\log z \\ &\text{subject to} && y + z \leq 5 + 3 = 8. \end{aligned}$$

The Lagrangian is

$$L(y, z, \lambda) = 2\sqrt{y} + 2\log z + \lambda(8 - y - z).$$

The first-order conditions are

$$\begin{aligned} 0 &= \frac{\partial L}{\partial y} = \frac{1}{\sqrt{y}} - \lambda = 0 \iff y = \frac{1}{\lambda^2}, \\ 0 &= \frac{\partial L}{\partial z} = \frac{2}{z} - \lambda = 0 \iff z = \frac{2}{\lambda}. \end{aligned}$$

Substituting into the feasibility constraint, we get

$$y + z = 8 \iff \frac{1}{\lambda^2} + \frac{2}{\lambda} = 8 \iff \lambda = \frac{1}{2}.$$

Therefore $x_2^A = y = 1/\lambda^2 = 4$ and $x_2^B = z = 2/\lambda = 4$.

(b) (10 points) Let the price be $p_1 = 1$ and $p_2 = p$. Compute the equilibrium price p and the allocation.

Solution: In a quasi-linear economy, we know from the lecture notes that the price is equal to the Lagrange multiplier. Therefore $p = \lambda = \frac{1}{2}$, and the consumption of good 2 is the same as above. By the budget constraint, the consumption of good 1 is $x_1^A = e_1^A + p(e_2^A - x_2^A) = \frac{9}{2}$ and $x_1^B = e_1^B + p(e_2^B - x_2^B) = \frac{19}{2}$.

6. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility functions are

$$\begin{aligned} u_A(x_1, x_2) &= \log x_1 + 2\log x_2, \\ u_B(x_1, x_2) &= 3x_1 + x_2. \end{aligned}$$

The endowments are $(e_1^A, e_2^A) = (6, 3)$ and $(e_1^B, e_2^B) = (10, 50)$.

(a) (3 points) What is the name of the type of agent A's utility function?

Solution: Cobb-Douglas.

(b) (5 points) Let the prices be $p_1 = 1$ and $p_2 = p$. Compute the demand of agent A.

> **Solution:** Using the Cobb-Douglas formula, we get
>
> $$(x_1^A, x_2^A) = \left(\frac{1}{3}\frac{6+3p}{1}, \frac{2}{3}\frac{6+3p}{p}\right) = \left(2+p, \frac{4+2p}{p}\right).$$

(c) (5 points) Assuming agent B consumes positive amounts of both goods, what is p?

> **Solution:** If agent B consumes positive amounts of both goods, then the Lagrangian is
>
> $$L(x_1, x_2, \lambda) = 3x_1 + x_2 + \lambda(w - x_1 - px_2),$$
>
> where w is the wealth of agent B. The first-order conditions are
>
> $$\frac{\partial L}{\partial x_1} = 0 \iff 3 - \lambda = 0,$$
> $$\frac{\partial L}{\partial x_2} = 0 \iff 1 - \lambda p = 0,$$
>
> so it must be $\lambda = 3$ and $p = \frac{1}{3}$.

(d) (7 points) Compute the competitive equilibrium.

> **Solution:** Substituting the price $p = 1/3$ into agent A's demand, his demand is
>
> $$(x_1^A, x_2^A) = \left(\frac{7}{3}, 14\right).$$
>
> By market clearing, the demand of agent B must be
>
> $$x_1^B = 6 + 10 - \frac{7}{3} = \frac{41}{3},$$
> $$x_2^B = 3 + 50 - 14 = 39.$$