# Mathematical Economics
# Midterm Exam 2

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Name: ______________________________________________

Instructions:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 80 minutes.
- This exam has 4 questions on 5 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Submit your entire exam sheet before leaving the room, even if some parts are empty or you intend to drop the class.

| Question: | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | 25 | 20 | 20 | 35 | 100 |
| Score: | | | | | |

1. Consider an economy with $L$ goods and $I$ agents. Agent $i$ has endowment $e_i$ and a locally nonsatiated utility function $u_i(x)$. Let $p$ be the price vector. If the notation bothers you, you may set $L = 2$ and $I = 2$.

   (a) (5 points) What is the definition of local nonsatiation? You may explain in words (maximum 4 points) or mathematically.

   > **Solution:** In words, a utility function is locally nonsatiated if for any bundle you can find an arbitrarily close bundle that gives higher utility. Mathematically, the utility function $u(x)$ is locally nonsatiated if for all $x$ and $\epsilon > 0$, there exists $y$ with $\|y - x\| < \epsilon$ such that $u(y) > u(x)$.

   (b) (5 points) Suppose $x_i$ solves the utility maximization problem

   $$\text{maximize } u_i(x) \text{ subject to } p \cdot x \leq p \cdot e_i.$$

   Explain why it must be the case that $p \cdot x_i = p \cdot e_i$.

   > **Solution:** If $p \cdot x_i < p \cdot e_i$, we can take a small ball with center $x_i$ and radius $\epsilon > 0$ that is contained in the budget set. By local nonsatiation, we can take a bundle $y$ in this ball such that $u_i(y) > u_i(x_i)$, which contradicts utility maximization. See lecture note for a rigorous proof.

   (c) (5 points) What does it mean that an allocation $(y_i)$ Pareto dominates the allocation $(x_i)$? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

   > **Solution:** $(y_i)$ Pareto dominates $(x_i)$ if $u_i(y_i) \geq u_i(x_i)$ for all $i$ and $u_i(y_i) > u_i(x_i)$ for some $i$.

   (d) (3 points) What does it mean that the feasible allocation $(x_i)$ is Pareto efficient? You can explain in words.

   > **Solution:** $(x_i)$ is Pareto efficient if no other feasible allocation $(y_i)$ Pareto dominates it.

   (e) (7 points) Let $\{p, (x_i)\}$ be an Arrow-Debreu equilibrium. Prove that $(x_i)$ is Pareto efficient.

   > **Solution:** See the lecture note for the proof of the First Welfare Theorem.

2. Consider an economy with two agents and two goods. The utility functions are

$$u_1(x_1, x_2) = \sqrt{x_1 x_2},$$
$$u_2(x_1, x_2) = \min\left\{\sqrt{x_1 x_2}, 7\right\}.$$

The initial endowments are $e_1 = (3, 12)$ and $e_2 = (12, 3)$.

(a) (5 points) Show that the price vector $(p_1, p_2) = (1, 1)$ and the allocation $x_1 = x_2 = (7.5, 7.5)$ constitute a competitive equilibrium.

**Solution:** Agent 1 has a Cobb-Douglas utility function, so with price $(1, 1)$ the demand is $(7.5, 7.5)$. Agent 2's utility is at most 7, and the bundle $(7.5, 7.5)$ gives utility 7 and is budget feasible. This allocation is feasible. Therefore it is a competitive equilibrium.

(b) (7 points) Show that the equilibrium allocation is Pareto inefficient.

**Solution:** The allocation $x_1 = (8, 8)$ and $x_2 = (7, 7)$ is feasible and makes agent 1 strictly better off, while making agent 2 indifferent. Therefore it Pareto improves the competitive equilibrium allocation, so the competitive equilibrium is Pareto inefficient.

(c) (8 points) Does this example contradict the first welfare theorem? Answer yes or no, and explain why.

**Solution:** No (4 points), because agent 2's utility function is not locally non-satiated (4 points).

3. Consider an economy with two countries, $i = A, B$, and two physical goods, $l = 1, 2$. The endowment is $e_A = (6, 1)$ and $e_B = (1, 6)$. The utility function is $u(x_1, x_2) = x_1 x_2$ for all agents. Suppose that there are transportation costs, and 50% of the exported goods perish by the time they reach the destination.

   (a) (5 points) How many kinds of goods are there in the world? Answer the number and explain the reason.

   **Solution:** 4, because there are two physical goods in two locations. (3 points for the answer, 2 points for the reason.)

   (b) (5 points) Assuming that country $A$ imports good 2, what is its price? (Set the price of good 1 equal to 1.)

   **Solution:** Relabel the goods by $l = 1, 2, 3, 4$, where good 1 is physical good 1 in country A, good 2 is physical good 2 in country A, and so on. Let $(p_1, p_2, p_3, p_4)$ be the price. By symmetry, the equilibrium price must satisfy $p_1 = p_4 = 1$ and $p_2 = p_3$. If country $B$ exports $e$ units of good 4, it becomes $0.5e$ units of good 2. By doing so, the profit is $(0.5p_2 - p_4)e$. By profit maximization, we get

   $$(0.5p_2 - p_4)e = 0 \iff p_2 = 2p_4 = 2.$$

(c) (10 points) Compute the free trade equilibrium.

**Solution:** By the Cobb-Douglas formula, the demand of country $A, B$ is

$$\left(\frac{6p_1 + p_2}{2p_1}, \frac{6p_1 + p_2}{2p_2}, 0, 0\right) = (4, 2, 0, 0),$$
$$\left(0, 0, \frac{p_3 + 6p_4}{2p_3}, \frac{p_3 + 6p_4}{2p_4}\right) = (0, 0, 2, 4).$$

Therefore the net import is $y_A = (-2, 1)$ for country $A$ and $y_B = (1, -2)$ for country $B$.

4. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, $A, B$. Agents 1 and 2 live in country $A$ and agent 3 lives in country $B$. The utility functions are

$$u_1(x_1, x_2) = x_1^2 x_2,$$
$$u_2(x_1, x_2) = x_1 x_2^2,$$
$$u_3(x_1, x_2) = x_1 x_2.$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (18, 6)$. In answering questions below, in order to make the notation consistent use $x_{il}$ for consumption of good $l$ by agent $i$. (So $x_{12}$ is consumption of good 2 by agent 1, for example.) Also, use $p_1 = 1$ and $p_2 = p$ for the prices.

(a) (5 points) Compute the competitive equilibrium when country $A$ is in autarky as well as the utility level of each agent.

**Solution:** Using the Cobb-Douglas formula, the demand of agents 1, 2 are

$$(x_{11}, x_{12}) = \left(\frac{2(3 + 3p)}{3}, \frac{3 + 3p}{3p}\right) = (2 + 2p, 1 + 1/p),$$
$$(x_{21}, x_{22}) = \left(\frac{3 + 3p}{3}, \frac{2(3 + 3p)}{3p}\right) = (1 + p, 2 + 2/p).$$

Therefore by market clearing we have

$$(2 + 2p) + (1 + p) = 3 + 3 \iff p = 1.$$

The demand is $(x_{11}, x_{12}) = (4, 2)$, $(x_{21}, x_{22}) = (2, 4)$. The utility level is $u_1^a = 4^2 \cdot 2 = 32$ and $u_2^a = 2 \cdot 4^2 = 32$.

(b) (10 points) Compute the free trade equilibrium price and allocation.

**Solution:** Using the Cobb-Douglas formula, the demand of agent 3 is

$$(x_{31}, x_{32}) = \left(\frac{18+6p}{2}, \frac{18+6p}{2p}\right) = (9+3p, 3+9/p).$$

Hence by market clearing, we have

$$(2+2p) + (1+p) + (9+3p) = 3+3+18 \iff p = 2.$$

Again using the Cobb-Douglas formula, the consumption of each agents are $(x_{11}, x_{12}) = (6, 3/2)$, $(x_{21}, x_{22}) = (3, 3)$, and $(x_{31}, x_{32}) = (15, 15/2)$.

(c) (5 points) Compute the utility level of each agent and determine who gained from trade and who lost.

**Solution:** Utility levels are

$$\begin{aligned}
u_1^f &= 6^2 \cdot 3/2 = 54 > 32 = u_1^a, \\
u_2^f &= 3 \cdot 3^2 = 27 < 32 = u_2^a, \\
u_3^f &= 15 \cdot 15/2 = \frac{225}{2} > 108 = 18 \cdot 6 = u_3^a.
\end{aligned}$$

Therefore agents 1 and 3 gained from trade and agent 2 lost.

(d) (15 points) Find a tax scheme in country $A$ such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

**Solution:** Consider a hypothetical economy in which agents start with the autarky allocation $e_1' = (4,2)$, $e_2' = (2,4)$, and $e_3' = (18,6)$. Using the Cobb-Douglas formula, the demand is

$$\begin{aligned}
(x_{11}, x_{12}) &= \left(\frac{2(4+2p)}{3}, \frac{4+2p}{3p}\right), \\
(x_{21}, x_{22}) &= \left(\frac{2+4p}{3}, \frac{2(2+4p)}{3p}\right), \\
(x_{31}, x_{32}) &= \left(\frac{18+6p}{2}, \frac{18+6p}{2p}\right).
\end{aligned}$$

The market clearing condition is

$$\frac{2(4+2p)}{3} + \frac{2+4p}{3} + \frac{18+6p}{2} = 4+2+18 \iff p = \frac{35}{17}.$$

This price should be the free trade price after the tax and transfer. To make the autarky equilibrium allocation just affordable, taxes should be set such that $p \cdot x_i^a = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i^a)$. Therefore

$$\begin{aligned} t_1 &= (1, 35/17) \cdot (3-4, 3-2) = -1 + \frac{35}{17} = \frac{18}{17}, \\ t_2 &= (1, 35/17) \cdot (3-2, 3-4) = 1 - \frac{35}{17} = -\frac{18}{17}. \end{aligned}$$

With these tax/transfers and price, since the autarky allocation is affordable, the equilibrium allocation Pareto dominates the autarky allocation. (5 points for the candidate price, 5 points for the tax/transfers, and 5 points for the explanation why it is Pareto improving.)