Mathematical Economics
Final Exam

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Name: ______________________________

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 180 minutes.
- This exam has 8 questions on 9 pages excluding the cover page, for a total of 110 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Use the notation x_{il} for agent i's consumption of good l.
- Unless otherwise stated, normalize the price of goods so that $p_1 = 1$.

Question:	1	2	3	4	5	6	7	8	Total
Points:	20	10	15	10	10	15	20	10	110
Score:									

1. (a) (2 points) Legibly write down your name in the designated area on the cover and carefully read the instructions.

   (b) (2 points) Suppose that there are two goods labeled by $l = 1, 2$. If your endowment is (e_1, e_2) and the price is (p_1, p_2), what is your wealth?

   (c) (2 points) What is the definition of a convex set?

   (d) (4 points) What is the statement of the Brouwer fixed point theorem?

   (e) (5 points) Suppose that you are an investment advisor. You know that the average risk tolerance in the economy is $\bar{\tau} = 1$ and your client has risk tolerance $\tau_i = 1.5$. Your client has 1 million dollars to invest. Assuming that the capital asset pricing model (CAPM) is true, what is your investment advice?

   (f) (5 points) Suppose that the expected market return is 8% and the risk-free rate is 2%. If an asset has $\beta = 0.5$, what is its expected return?

2. (10 points) Consider an economy with two agents ($i = 1, 2$) and two goods ($l = 1, 2$).

The utility functions are

$$u_1(x_1, x_2) = x_1 - \frac{9}{x_2},$$
$$u_2(x_1, x_2) = x_1 + 2 \log x_2.$$

The initial endowments are $e_1 = (5, 1)$ and $e_2 = (4, 4)$. Compute the competitive equilibrium.

3. Consider an economy with I agents with utility functions (u_i). An *allocation* is denoted by (x_i), where x_i is the consumption bundle of agent i. The initial endowment is (e_i).

   (a) (5 points) What does it mean that the allocation (x_i) is Pareto efficient? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

   (b) (5 points) What does the first welfare theorem say? You can explain in words.

(c) (5 points) How are the assumptions of the first and second welfare theorems different? You can explain in words.

4. Consider an economy with two agents and two goods. The utility functions are

$$u_1(x_1, x_2) = \sqrt{x_1 x_2},$$
$$u_2(x_1, x_2) = \min\left\{\sqrt{x_1 x_2}, 4\right\}.$$

The initial endowments are $e_1 = (1, 9)$ and $e_2 = (9, 1)$.

(a) (3 points) Show that the price vector $(p_1, p_2) = (1, 1)$ and the allocation $x_1 = x_2 = (5, 5)$ constitute a competitive equilibrium.

(b) (3 points) Show that the equilibrium allocation is Pareto inefficient.

(c) (4 points) Does this example contradict the first welfare theorem? Answer yes or no, and explain why.

5. Consider an economy with two countries, $i = A, B$, and two physical goods, $l = 1, 2$. The endowment is $e_A = (10, 1)$ and $e_B = (1, 10)$. The utility function is $u(x_1, x_2) = x_1 x_2$ for all agents. Suppose that there are transportation costs, and 50% of the exported goods perish by the time they reach the destination.

   (a) (2 points) How many kinds of goods are there in the world? Answer the number and explain the reason.

   (b) (3 points) Assuming that country A imports good 2, what is its price? (Set the price of good 1 equal to 1.)

   (c) (5 points) Compute the free trade equilibrium.

6. Consider an economy with two countries, A, B, and two goods, $l = 1, 2$. There are many agents, and the utility function of agent i is $u_i(x_1, x_2)$, which is increasing, quasi-concave, and differentiable. Let the world price of good 2 equal p.

   (a) (5 points) If all countries adopt free trade, what is the marginal rate of substitution between goods 1 and 2 evaluated at the equilibrium allocation?

   (b) (5 points) Suppose the government of country A is concerned about protecting industry 2 and imposes a tariff, so the domestic price of good 2 in country A is $p_2 = p(1 + \tau)$, where $\tau > 0$ is tariff. If country B adopts free trade, prove that no matter how the government of country A transfers the revenue from tariff to its citizens, the equilibrium is Pareto inefficient.

(c) (5 points) Suppose that you are an economist advising the government for trade policy. Propose a policy that achieves Pareto efficiency but at the same time makes everybody at least as well off as autarky.

7. Consider an economy with two agents ($i = 1, 2$), two periods ($t = 0, 1$), and two assets

(stock and bond). The bond is in zero net supply. There are two states ($s = u, d$, for up and down) of the world at $t = 1$ and the stock return is

$$\frac{S_1}{S_0} = \begin{cases} \mu + \sigma, & \text{(with probability 1/2)} \\ \mu - \sigma, & \text{(with probability 1/2)} \end{cases}$$

where μ is the expected return and $\sigma > 0$ is volatility. Agent i has initial wealth w_i and wants to maximize

$$v_i(\theta) = \mathrm{E}[R(\theta)] - \frac{1}{2\tau_i}\mathrm{Var}[R(\theta)],$$

where θ is the fraction of wealth invested in the stock, $\tau_i > 0$ is the risk tolerance,

$$R(\theta) = \frac{S_1}{S_0}\theta + R_f(1 - \theta)$$

is the portfolio return, and R_f is the gross risk-free rate (to be determined in equilibrium).

(a) (4 points) Express agent i's optimal portfolio, θ_i, using τ_i, μ, σ, R_f.

(b) (4 points) Assuming $w_1 + w_2 = 1$, compute the equilibrium risk-free rate R_f.

(c) (4 points) Let p_s be the price of a security that pays 1 in state s ($s = u, d$) and zero otherwise. Write down two equations that p_u, p_d satisfy.

(d) (8 points) Compute the $t = 0$ price of a put option with strike price K, which is a contract that pays $\max\{0, K - S_1\}$ at $t = 1$. The answer should contain only S_0, K, μ, σ, and $\bar{\tau} = w_1\tau_1 + w_2\tau_2$.

8. (10 points) (extra credit) A course (not this course) has two midterms and a final exam. The raw scores are denoted by M_1, M_2, F, which can take continuous values between 0 and 100. Since the raw scores were somewhat disappointing, the professor

decided to replace them by

$$
\begin{aligned}
M_1' &= \frac{5}{6}\sqrt{M_1 + 41}, \\
M_2' &= \frac{1}{2}\max\left\{M_1 - 7, 2M_2 - \frac{1}{100}M_2^2\right\} + \frac{1}{3}M_2 + \frac{1}{6}M_1, \\
F' &= 10 + 4\sqrt{\max\left\{M_2, F\right\}} + \frac{1}{3}F + \frac{1}{6}\min\left\{M_1, F\right\}.
\end{aligned}
$$

Prove that there exist raw scores (M_1, M_2, F) that remain unchanged under this transformation.