Mathematical Economics
Final Exam

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Name: ______________________________

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 180 minutes.
- This exam has 8 questions on 8 pages excluding the cover page, for a total of 110 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Use the notation x_{il} for agent i's consumption of good l.
- Unless otherwise stated, normalize the price of goods so that $p_1 = 1$.

Question:	1	2	3	4	5	6	7	8	Total
Points:	20	10	15	10	10	15	20	10	110
Score:									

1. (a) (2 points) Legibly write down your name in the designated area on the cover and carefully read the instructions.

   (b) (2 points) Suppose that there are two goods labeled by $l = 1, 2$. If your endowment is (e_1, e_2) and the price is (p_1, p_2), what is your wealth?

   > **Solution:** $p_1 e_1 + p_2 e_2$.

   (c) (2 points) What is the definition of a convex set?

   > **Solution:** C is convex if for all $x, y \in C$ and $0 \le \alpha \le 1$, we have $(1-\alpha)x + \alpha y \in C$.

   (d) (4 points) What is the statement of the Brouwer fixed point theorem?

   > **Solution:** If C is nonempty, compact (closed and bounded), convex, and $f : C \to C$ is continuous, then there exists $x \in C$ such that $f(x) = x$. (1 point for saying that there is a fixed point, 1 point each for mentioning compactness, convexity, and continuity.)

   (e) (5 points) Suppose that you are an investment advisor. You know that the average risk tolerance in the economy is $\bar{\tau} = 1$ and your client has risk tolerance $\tau_i = 1.5$. Your client has 1 million dollars to invest. Assuming that the capital asset pricing model (CAPM) is true, what is your investment advice?

   > **Solution:** The client should invest fraction of wealth $\tau_i/\bar{\tau} = 1.5$ in the market portfolio, and $1 - \tau_i/\bar{\tau} = -0.5$ in the risk-free asset. Therefore the client should borrow 0.5 million and invest 1.5 million in the stock market. (2 points for mentioning the market portfolio, 1 point for mentioning the risk-free asset, and 2 points for the correct numbers.)

   (f) (5 points) Suppose that the expected market return is 8% and the risk-free rate is 2%. If an asset has $\beta = 0.5$, what is its expected return?

   > **Solution:** By the covariance pricing formula,
   >
   > $$\mathrm{E}[R_j] - R_f = \beta_j(\mathrm{E}[R_m] - R_f).$$
   >
   > Therefore the expected return is $1.02 + 0.5 \times (1.08 - 1.02) = 1.05$, so 5%. (2 points for the correct answer, 3 points for derivation.)

2. (10 points) Consider an economy with two agents ($i = 1, 2$) and two goods ($l = 1, 2$). The utility functions are

$$u_1(x_1, x_2) = x_1 - \frac{9}{x_2},$$
$$u_2(x_1, x_2) = x_1 + 2 \log x_2.$$

The initial endowments are $e_1 = (5, 1)$ and $e_2 = (4, 4)$. Compute the competitive equilibrium.

Solution: Since the utility functions are quasi-linear, it suffices to maximize the sum of the nonlinear part. To simplify notations, let $x_{12} = y$ and $x_{22} = z$. Then the problem of maximizing the sum of the nonlinear part becomes

$$\begin{aligned} &\text{maximize} && -\frac{9}{y} + 2\log z \\ &\text{subject to} && y + z \leq 1 + 4 = 5. \end{aligned}$$

The Lagrangian is

$$L(y, z, p) = -\frac{9}{y} + 2\log z + p(5 - y - z),$$

where p is the Lagrange multiplier (which is the price of good 2). The first-order conditions are $9/y^2 = p$ and $2/z = p$, so $y = 3/\sqrt{p}$ and $z = 2/p$. Substituting into the feasibility constraint, we get

$$y + z = 5 \iff \frac{3}{\sqrt{p}} + \frac{2}{p} = 5.$$

This is a quadratic equation in $1/\sqrt{p}$, so you can solve. The solution is $p = 1$. Therefore $x_{12} = y = 3$ and $x_{22} = z = 2$. By the budget constraint, the competitive equilibrium is

$$\{(p_1, p_2), (x_{11}, x_{12}), (x_{21}, x_{22})\} = \{(1, 1), (3, 3), (6, 2)\}.$$

3. Consider an economy with I agents with utility functions (u_i). An *allocation* is denoted by (x_i), where x_i is the consumption bundle of agent i. The initial endowment is (e_i).

   (a) (5 points) What does it mean that the allocation (x_i) is Pareto efficient? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

   Solution: (x_i) is Pareto efficient if

   1. (x_i) is feasible, so $\sum_{i=1}^I x_i \leq \sum_{i=1}^I e_i$, and
   2. there are no other feasible allocation (y_i) that Pareto dominate (x_i). (y_i) Pareto dominates (x_i) if $u_i(y_i) \geq u_i(x_i)$ for all i and $u_i(y_i) > u_i(x_i)$ for some i.

In words, an allocation is Pareto efficient if it is impossible to make somebody better off without making somebody worse off.

(b) (5 points) What does the first welfare theorem say? You can explain in words.

Solution: The first welfare theorem says that the competitive equilibrium allocation is Pareto efficient, provided that utility functions are locally non-satiated. (1 point for referring to the equilibrium allocation, 3 points for saying Pareto efficient, and 1 point for mentioning local non-satiation.)

(c) (5 points) How are the assumptions of the first and second welfare theorems different? You can explain in words.

Solution: For the first welfare theorem, the only assumption is local non-satiation (1 point). For the second welfare theorem, we assume that utility functions are continuous, quasi-concave, locally non-satiated, and that the allocation (x_i) is interior—that is, $x_i \gg 0$ for all i (1 point for each of these four assumptions).

4. Consider an economy with two agents and two goods. The utility functions are

$$u_1(x_1, x_2) = \sqrt{x_1 x_2},$$
$$u_2(x_1, x_2) = \min\left\{\sqrt{x_1 x_2}, 4\right\}.$$

The initial endowments are $e_1 = (1, 9)$ and $e_2 = (9, 1)$.

(a) (3 points) Show that the price vector $(p_1, p_2) = (1, 1)$ and the allocation $x_1 = x_2 = (5, 5)$ constitute a competitive equilibrium.

Solution: Agent 1 has a Cobb-Douglas utility function, so with price $(1, 1)$ the demand is $(5, 5)$. Agent 2's utility is at most 4, and the bundle $(5, 5)$ gives utility 4 and is budget feasible. This allocation is feasible. Therefore it is a competitive equilibrium.

(b) (3 points) Show that the equilibrium allocation is Pareto inefficient.

Solution: The allocation $x_1 = (6, 6)$ and $x_2 = (4, 4)$ is feasible and makes agent 1 strictly better off, while making agent 2 indifferent. Therefore it Pareto improves the competitive equilibrium allocation, so the competitive equilibrium is Pareto inefficient.

(c) (4 points) Does this example contradict the first welfare theorem? Answer yes or no, and explain why.

Solution: No (2 points), because agent 2's utility function is not locally non-satiated (2 points).

5. Consider an economy with two countries, $i = A, B$, and two physical goods, $l = 1, 2$. The endowment is $e_A = (10, 1)$ and $e_B = (1, 10)$. The utility function is $u(x_1, x_2) = x_1 x_2$ for all agents. Suppose that there are transportation costs, and 50% of the exported goods perish by the time they reach the destination.

   (a) (2 points) How many kinds of goods are there in the world? Answer the number and explain the reason.

   Solution: 4, because there are two physical goods in two locations. (1 point for the answer, 1 point for the reason.)

   (b) (3 points) Assuming that country A imports good 2, what is its price? (Set the price of good 1 equal to 1.)

   Solution: Relabel the goods by $l = 1, 2, 3, 4$, where good 1 is physical good 1 in country A, good 2 is physical good 2 in country A, and so on. Let (p_1, p_2, p_3, p_4) be the price. By symmetry, the equilibrium price must satisfy $p_1 = p_4 = 1$ and $p_2 = p_3$. If country B exports e units of good 4, it because $0.5e$ units of good 2. By doing so, the profit is $(0.5p_2 - p_4)e$. By profit maximization, we get

   $$(0.5p_2 - p_4)e = 0 \iff p_2 = 2p_4 = 2.$$

   (c) (5 points) Compute the free trade equilibrium.

   Solution: By the Cobb-Douglas formula, the demand of country A, B is

   $$\left(\frac{10p_1 + p_2}{2p_1}, \frac{10p_1 + p_2}{2p_2}, 0, 0\right) = (6, 3, 0, 0),$$
   $$\left(0, 0, \frac{p_3 + 10p_4}{2p_3}, \frac{p_3 + 10p_4}{2p_4}\right) = (0, 0, 3, 6).$$

   Therefore the net import is $y_A = (-4, 2)$ for country A and $y_B = (2, -4)$ for country B.

6. Consider an economy with two countries, A, B, and two goods, $l = 1, 2$. There are many agents, and the utility function of agent i is $u_i(x_1, x_2)$, which is increasing, quasi-concave, and differentiable. Let the world price of good 2 equal p.

   (a) (5 points) If all countries adopt free trade, what is the marginal rate of substitution between goods 1 and 2 evaluated at the equilibrium allocation?

Solution: By the first-order condition ($\frac{\partial u_i}{\partial x_l} = \lambda_i p_l$), the marginal rate of substitution at the equilibrium allocation is

$$\frac{\partial u_i}{\partial x_2} \Big/ \frac{\partial u_i}{\partial x_1} = p.$$

(b) (5 points) Suppose the government of country A is concerned about protecting industry 2 and imposes a tariff, so the domestic price of good 2 in country A is $p_2 = p(1+\tau)$, where $\tau > 0$ is tariff. If country B adopts free trade, prove that no matter how the government of country A transfers the revenue from tariff to its citizens, the equilibrium is Pareto inefficient.

Solution: Due to the tariff, the marginal rate of substitution of an agent at the equilibrium allocation in contry A is

$$\frac{\partial u_i}{\partial x_2} \Big/ \frac{\partial u_i}{\partial x_1} = p(1+\tau).$$

Since this number is different from the marginal rate of substitution of an agent in country B (which is p), the equilibrium is inefficient.

(c) (5 points) Suppose that you are an economist advising the government for trade policy. Propose a policy that achieves Pareto efficiency but at the same time makes everybody at least as well off as autarky.

Solution: According to the lecture note, one way to achieve both goals is to adopt a free trade policy but transfer wealth across citizens so that the autarky allocation is just affordable.

7. Consider an economy with two agents ($i = 1, 2$), two periods ($t = 0, 1$), and two assets (stock and bond). The bond is in zero net supply. There are two states ($s = u, d$, for up and down) of the world at $t = 1$ and the stock return is

$$\frac{S_1}{S_0} = \begin{cases} \mu + \sigma, & \text{(with probability 1/2)} \\ \mu - \sigma, & \text{(with probability 1/2)} \end{cases}$$

where μ is the expected return and $\sigma > 0$ is volatility. Agent i has initial wealth w_i and wants to maximize

$$v_i(\theta) = \mathrm{E}[R(\theta)] - \frac{1}{2\tau_i} \mathrm{Var}[R(\theta)],$$

where θ is the fraction of wealth invested in the stock, $\tau_i > 0$ is the risk tolerance,

$$R(\theta) = \frac{S_1}{S_0}\theta + R_f(1-\theta)$$

is the portfolio return, and R_f is the gross risk-free rate (to be determined in equilibrium).

(a) (4 points) Express agent i's optimal portfolio, θ_i, using τ_i, μ, σ, R_f.

> **Solution:** The expected return is $\mathrm{E}[R(\theta)] = R_f + (\mu - R_f)\theta$ and the variance is $\mathrm{Var}[R(\theta)] = \sigma^2\theta^2$. Therefore
>
> $$v_i(\theta) = R_f + (\mu - R_f)\theta - \frac{1}{2\tau_i}\sigma^2\theta^2.$$
>
> By the first-order condition, we get
>
> $$v_i'(\theta) = 0 \iff \mu - R_f - \frac{\sigma^2}{\tau_i}\theta = 0 \iff \theta_i = \tau_i\frac{\mu - R_f}{\sigma^2}.$$
>
> (2 points for getting $v_i(\theta)$, 1 point for the first-order condition, 1 point for the answer.)

(b) (4 points) Assuming $w_1 + w_2 = 1$, compute the equilibrium risk-free rate R_f.

> **Solution:** By market clearing, we have
>
> $$w_1(1-\theta_1) + w_2(1-\theta_2) = 0 \iff R_f = \mu - \frac{\sigma^2}{w_1\tau_1 + w_2\tau_2}.$$
>
> (2 points for the market clearing equation, 2 points for the answer.)

(c) (4 points) Let p_s be the price of a security that pays 1 in state s ($s = u, d$) and zero otherwise. Write down two equations that p_u, p_d satisfy.

> **Solution:** Since the risk-free asset pays 1 no matter what, holding the risk-free asset is the same as holding 1 unit of security u and 1 unit of security d. Since the price of either one must be the same, we get
>
> $$\frac{1}{R_f} = p_u 1 + p_d 1.$$
>
> Similarly, for the stock we have
>
> $$S_0 = p_u S_u + p_d S_d = p_u(\mu + \sigma)S_0 + p_d(\mu - \sigma)S_0.$$
>
> (2 points for the idea, 1 point each for the equations.)

(d) (8 points) Compute the $t = 0$ price of a put option with strike price K, which is a contract that pays $\max\{0, K - S_1\}$ at $t = 1$. The answer should contain only S_0, K, μ, σ, and $\bar{\tau} = w_1\tau_1 + w_2\tau_2$.

Solution: Canceling S_0 and writing in matrix form, we get

$$\begin{aligned}&\begin{bmatrix}\mu+\sigma & \mu-\sigma\\ 1 & 1\end{bmatrix}\begin{bmatrix}p_u\\ p_d\end{bmatrix}=\begin{bmatrix}1\\ 1/R_f\end{bmatrix}\\ \iff &\begin{bmatrix}p_u\\ p_d\end{bmatrix}=\frac{1}{2\sigma}\begin{bmatrix}1 & -\mu+\sigma\\ -1 & \mu+\sigma\end{bmatrix}\begin{bmatrix}1\\ 1/R_f\end{bmatrix}=\frac{1}{2(\mu\bar{r}-\sigma^2)}\begin{bmatrix}\bar{r}-\sigma\\ \bar{r}+\sigma\end{bmatrix}.\end{aligned}$$

Therefore the price of the put option is

$$\begin{aligned}&p_u\max\{0,K-S_u\}+p_d\max\{0,K-S_d\}\\ &=\frac{(\bar{r}-\sigma)\max\{0,K-S_0(\mu+\sigma)\}+(\bar{r}+\sigma)\max\{0,K-S_0(\mu-\sigma)\}}{2(\mu\bar{r}-\sigma^2)}\\ &=\begin{cases}0, & (K<S_0(\mu-\sigma))\\ \frac{(\bar{r}+\sigma)(K-S_0(\mu-\sigma))}{2(\mu\bar{r}-\sigma^2)}, & (S_0(\mu-\sigma)\le K<S_0(\mu+\sigma))\\ \frac{K}{R_f}-S_0. & (S_0(\mu+\sigma)\le K)\end{cases}\end{aligned}$$

(3 points for solving for p_u, p_d. 2 points for the idea of how to price the call option, and 1 point each for the correct price in each case.)

8. (10 points) (extra credit) A course (not this course) has two midterms and a final exam. The raw scores are denoted by M_1, M_2, F, which can take continuous values between 0 and 100. Since the raw scores were somewhat disappointing, the professor decided to replace them by

$$\begin{aligned}M_1'&=\frac{5}{6}\sqrt{M_1+41},\\ M_2'&=\frac{1}{2}\max\left\{M_1-7,2M_2-\frac{1}{100}M_2^2\right\}+\frac{1}{3}M_2+\frac{1}{6}M_1,\\ F'&=10+4\sqrt{\max\{M_2,F\}}+\frac{1}{3}F+\frac{1}{6}\min\{M_1,F\}.\end{aligned}$$

Prove that there exist raw scores (M_1, M_2, F) that remain unchanged under this transformation.

Solution: Clearly $M_1', M_2', F' \geq 0$. Since $M_1 \leq 100$, we have

$$M_1'=\frac{5}{6}\sqrt{M_1+41}\le\frac{5}{6}\sqrt{141}\le\frac{5}{6}\sqrt{144}=100.$$

Similarly, you can show $M_2', F' \leq 100$. Since the set $[0,100]\times[0,100]\times[0,100]$ is nonempty, compact, convex, and the score transformation is continuous, by the Brouwer fixed point theorem there is a fixed point. (2 points each for showing that each of M_1', M_2', F' stays within $[0,100]$. 1 point for mentioning Brouwer. 1

point each for verifying (stating) compactness (closed and bounded), convexity, and continuity.)