Mathematical Economics
Midterm Exam 1

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Name: ____________________

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 80 minutes.
- This exam has 5 questions on 5 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Submit your entire exam sheet before leaving the room, even if some parts are empty or you intend to drop the class.

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

1. (a) (5 points) Suppose that there are 2 goods, an agent has endowment $e = (e_1, e_2)$, and the price vector is $p = (p_1, p_2)$. Define the budget set of the agent.

   (b) (5 points) What is the definition of local nonsatiation?

   (c) (5 points) What is the definition of a convex set?

   (d) (5 points) Prove that the budget set is convex.

2. Consider an agent with a Cobb-Douglas utility function

$$u(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2,$$

   where $0 < \alpha < 1$. Assume that the endowment of the agent is (e_1, e_2) and the price of each good is $p_1 = 1$ and $p_2 = p > 0$.

   (a) (5 points) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$. You may use $w = e_1 + pe_2$ for the wealth.

   (b) (5 points) Solve for the demand (x_1, x_2).

(c) (5 points) In general, the locus of demand when the price changes (that is, the set of (x_1, x_2) corresponding to each $0 < p < \infty$) is called the *offer curve*. Express the offer curve of the agent as $x_2 = g(x_1)$, where g is some function. (Hint: g depends on the parameters α, e_1, e_2, but not on p.)

(d) (5 points) Draw a picture of the offer curve on the x_1x_2 plane. Make sure to indicate the asymptotes, if there are any.

3. Consider an agent with a CES (constant elasticity of substitution) utility function

$$u(x_1, x_2) = \frac{1}{1-\sigma}\left(\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma}\right),$$

where $\alpha_1, \alpha_2 > 0$ and $1 \neq \sigma > 0$. Assume that the agent has wealth $w > 0$ and that the price each good is p_1, p_2.

(a) (5 points) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$.

(b) (5 points) Derive the first order conditions with respect to x_1 and x_2.

(c) (5 points) Express $\frac{x_2}{x_1}$ as a function of $\frac{p_2}{p_1}$.

(d) (5 points) In general,

$$\varepsilon = -\frac{\mathrm{d}\log(x_2/x_1)}{\mathrm{d}\log(p_2/p_1)}$$

is called the *elasticity of substitution* between good 1 and 2. (If you don't like the expression $\mathrm{d}\log(p_2/p_1)$, just set $t = \log(p_2/p_1)$ and think of $\log(x_2/x_1)$ as a function of t alone. "d" here refers to the derivative, of course.)

Compute the elasticity of substitution for an agent with CES utility.

4. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility function of each agent is

$$u_A(x_1, x_2) = x_1 + 2\sqrt{x_2},$$
$$u_B(x_1, x_2) = x_1 + \log x_2.$$

Suppose that the initial endowment is $(e_1^A, e_2^A) = (e_1^B, e_2^B) = (3, 3)$.

(a) (3 points) What is the name of this type of utility functions?

(b) (5 points) Consider the problem of maximizing the sum of utilities,

$$\begin{array}{ll} \text{maximize} & u_A(x_1^A, x_2^A) + u_B(x_1^B, x_2^B) \\ \text{subject to} & x_1^A + x_1^B \le e_1^A + e_1^B, \\ & x_2^A + x_2^B \le e_2^A + e_2^B. \end{array}$$

Let $(x_1^A, x_2^A, x_1^B, x_2^B)$ be a solution. Compute x_2^A and x_2^B.

(c) (5 points) Let the price be $p_1 = 1$ and $p_2 = p$. Compute the equilibrium price p and the allocation.

(d) (7 points) In general, consider an agent with wealth $w > 0$ and utility function

$$u(x_0, x_1, x_2) = x_0 + \phi_1(x_1) + \phi_2(x_2),$$

where ϕ_1, ϕ_2 are some functions. (You may assume they are increasing, differentiable, and concave.) If the price vector is $(1, p_1, p_2)$, show that the demand of good 1 depends only on p_1 and the demand of good 2 depends only on p_2.

5. Consider an economy with two periods denoted by $t = 0, 1$ (today and tomorrow). There is a single good at each period. As explained in the lecture, goods are distinguished by time, so introducing time does not change the model (at least formally).

The good is perishable, like raw fish, so the good today cannot be transformed into the good tomorrow.

(a) (5 points) Let p_0, p_1 be the price of 1 unit of good at each period, so actually p_1 is the *future price* (the amount of money you have to pay today in order to be delivered 1 unit of good tomorrow). Let the interest rate be $100r\%$. By arguing how much good you will get tomorrow by reducing the consumption of 1 unit of good today, express $1 + r$ as a function of p_0 and p_1.

(b) (5 points) Suppose that there are two agents, A and B, and the endowment of each agent is $e^A = e^B = (e_0, e_1)$. (So A and B go fishing and fish e_0 at $t = 0$ and e_1 at $t = 1$.) Each agent has the utility function

$$u^A(x_0, x_1) = (1 - \beta^A) \log x_0 + \beta^A \log x_1,$$
$$u^B(x_0, x_1) = (1 - \beta^B) \log x_0 + \beta^B \log x_1.$$

Let the price be $p_0 = 1$ and $p_1 = p$. Compute the demand of agent A, given the price p.

(c) (5 points) Compute the equilibrium price and interest rate.

(d) (5 points) Assume $\beta^A > \beta^B$. Which agent is saving? Which agent is borrowing?