Mathematical Economics
Midterm Exam 1

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Name: ______________________________

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 80 minutes.
- This exam has 5 questions on 6 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Submit your entire exam sheet before leaving the room, even if some parts are empty or you intend to drop the class.

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

1. (a) (5 points) Suppose that there are 2 goods, an agent has endowment $e = (e_1, e_2)$, and the price vector is $p = (p_1, p_2)$. Define the budget set of the agent.

Solution: $B(p) = \{x \in \mathbb{R}^2_+ \mid p \cdot x \leq p \cdot e\}$.

(b) (5 points) What is the definition of local nonsatiation?

Solution: $u(x)$ is locally nonsatiated if for all $\epsilon > 0$ and $x \in \mathbb{R}^L_+$, there exists $y \in \mathbb{R}^L_+$ such that $\|y - x\| < \epsilon$ and $u(y) > u(x)$.

(c) (5 points) What is the definition of a convex set?

Solution: C is convex if for all $x, y \in C$ and $\alpha \in [0, 1]$, we have $(1 - \alpha)x + \alpha y \in C$.

(d) (5 points) Prove that the budget set is convex.

Solution: If $x, y \in B(p)$, then $p \cdot x \leq p \cdot e$ and $p \cdot y \leq p \cdot e$. Letting $z = (1 - \alpha)x + \alpha y$, we obtain

$$p \cdot z = p \cdot ((1 - \alpha)x + \alpha y) = (1 - \alpha)p \cdot x + \alpha p \cdot y \leq p \cdot e,$$

so $B(p)$ is convex.

2. Consider an agent with a Cobb-Douglas utility function

$$u(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2,$$

where $0 < \alpha < 1$. Assume that the endowment of the agent is (e_1, e_2) and the price of each good is $p_1 = 1$ and $p_2 = p > 0$.

(a) (5 points) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$. You may use $w = e_1 + pe_2$ for the wealth.

Solution:

$$L(x_1, x_2, \lambda) = \alpha \log x_1 + (1 - \alpha) \log x_2 + \lambda(w - x_1 - px_2).$$

(b) (5 points) Solve for the demand (x_1, x_2).

Solution: $x_1 = \frac{\alpha w}{p_1} = \alpha(e_1 + pe_2)$, $x_2 = \frac{(1-\alpha)w}{p_2} = \frac{(1-\alpha)(e_1+pe_2)}{p}$.

(c) (5 points) In general, the locus of demand when the price changes (that is, the set of (x_1, x_2) corresponding to each $0 < p < \infty$) is called the *offer curve*. Express the offer curve of the agent as $x_2 = g(x_1)$, where g is some function. (Hint: g depends on the parameters α, e_1, e_2, but not on p.)

Solution: Solving for p as a function of x_1, we get $p = \frac{x_1 - \alpha e_1}{\alpha e_2}$ and $x_1 > \alpha e_1$. Substituting this into the formula for x_2 above, we get

$$x_2 = \frac{(1-\alpha)e_2 x_1}{x_1 - \alpha e_1} = (1-\alpha)e_2 + \frac{\alpha(1-\alpha)e_1 e_2}{x_1 - \alpha e_1}.$$

(d) (5 points) Draw a picture of the offer curve on the $x_1 x_2$ plane. Make sure to indicate the asymptotes, if there are any.

Solution: The above equation is a hyperbola with asymptotes $x_1 = \alpha e_1$ and $x_2 = (1-\alpha)e_2$, and it passes through the point (e_1, e_2).

3. Consider an agent with a CES (constant elasticity of substitution) utility function

$$u(x_1, x_2) = \frac{1}{1-\sigma}\left(\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma}\right),$$

where $\alpha_1, \alpha_2 > 0$ and $1 \neq \sigma > 0$. Assume that the agent has wealth $w > 0$ and that the price each good is p_1, p_2.

(a) (5 points) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$.

Solution:

$$L(x_1, x_2, \lambda) = \frac{1}{1-\sigma}\left(\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma}\right) + \lambda(w - p_1 x_1 - p_2 x_2).$$

(b) (5 points) Derive the first order conditions with respect to x_1 and x_2.

Solution: Taking the partial derivative of L with respect to x_1 and x_2 and setting them equal to 0, we get

$$\begin{aligned} \alpha_1 x_1^{-\sigma} &= \lambda p_1, \\ \alpha_2 x_2^{-\sigma} &= \lambda p_2. \end{aligned}$$

(c) (5 points) Express $\frac{x_2}{x_1}$ as a function of $\frac{p_2}{p_1}$.

Solution: Taking the ratio of the first order conditions, we get

$$\frac{\alpha_2}{\alpha_1}\left(\frac{x_2}{x_1}\right)^{-\sigma} = \frac{p_2}{p_1} \iff \frac{x_2}{x_1} = \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\sigma}}\left(\frac{p_2}{p_1}\right)^{-\frac{1}{\sigma}}.$$

(d) (5 points) In general,

$$\varepsilon = -\frac{\mathrm{d}\log(x_2/x_1)}{\mathrm{d}\log(p_2/p_1)}$$

is called the *elasticity of substitution* between good 1 and 2. (If you don't like the expression $\mathrm{d}\log(p_2/p_1)$, just set $t = \log(p_2/p_1)$ and think of $\log(x_2/x_1)$ as a function of t alone. "d" here refers to the derivative, of course.)

Compute the elasticity of substitution for an agent with CES utility.

Solution: Taking the logarithm of the above equation, we get

$$\log\frac{x_2}{x_1} = \frac{1}{\sigma}\log\frac{\alpha_2}{\alpha_1} - \frac{1}{\sigma}\log\frac{p_2}{p_1}.$$

Setting $t = \log(p_2/p_1)$, putting a minus sign, and differentiating with respect to t, we get

$$\varepsilon = -\frac{\mathrm{d}\log(x_2/x_1)}{\mathrm{d}\log(p_2/p_1)} = \frac{1}{\sigma}.$$

(Now you know why it's called CES!)

4. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility function of each agent is

$$u_A(x_1, x_2) = x_1 + 2\sqrt{x_2},$$
$$u_B(x_1, x_2) = x_1 + \log x_2.$$

Suppose that the initial endowment is $(e_1^A, e_2^A) = (e_1^B, e_2^B) = (3, 3)$.

(a) (3 points) What is the name of this type of utility functions?

Solution: Quasi-linear.

(b) (5 points) Consider the problem of maximizing the sum of utilities,

$$\begin{aligned} &\text{maximize} && u_A(x_1^A, x_2^A) + u_B(x_1^B, x_2^B) \\ &\text{subject to} && x_1^A + x_1^B \le e_1^A + e_1^B, \\ &&& x_2^A + x_2^B \le e_2^A + e_2^B. \end{aligned}$$

Let $(x_1^A, x_2^A, x_1^B, x_2^B)$ be a solution. Compute x_2^A and x_2^B.

Solution: Since the utility functions are quasi-linear, it suffices to maximize the sum of the nonlinear part. For notational simplicity, let $x_2^A = y$ and $x_2^B = z$. Then the problem becomes

$$\begin{aligned} &\text{maximize} && 2\sqrt{y} + \log z \\ &\text{subject to} && y + z \le 3 + 3 = 6. \end{aligned}$$

The Lagrangian is

$$L(y, z, \lambda) = 2\sqrt{y} + \log z + \lambda(6 - y - z).$$

The first-order conditions are

$$0 = \frac{\partial L}{\partial y} = \frac{1}{\sqrt{y}} - \lambda = 0 \iff y = \frac{1}{\lambda^2},$$
$$0 = \frac{\partial L}{\partial z} = \frac{1}{z} - \lambda = 0 \iff z = \frac{1}{\lambda}.$$

Substituting into the feasibility constraint, we get

$$y + z = 6 \iff \frac{1}{\lambda^2} + \frac{1}{\lambda} = 6 \iff \lambda = \frac{1}{2}.$$

Therefore $x_2^A = y = 1/\lambda^2 = 4$ and $x_2^B = z = 1/\lambda = 2$.

(c) (5 points) Let the price be $p_1 = 1$ and $p_2 = p$. Compute the equilibrium price p and the allocation.

Solution: In a quasi-linear economy, we know from the lecture note that the price is equal to the Lagrange multiplier. Therefore $p = \lambda = \frac{1}{2}$, and the consumption of good 2 is the same as above. By the budget constraint, the consumption of good 1 is $x_1^A = e_1^A + p(e_2^A - x_2^A) = \frac{5}{2}$ and $x_1^B = e_1^B + p(e_2^B - x_2^B) = \frac{7}{2}$.

(d) (7 points) In general, consider an agent with wealth $w > 0$ and utility function

$$u(x_0, x_1, x_2) = x_0 + \phi_1(x_1) + \phi_2(x_2),$$

where ϕ_1, ϕ_2 are some functions. (You may assume they are increasing, differentiable, and concave.) If the price vector is $(1, p_1, p_2)$, show that the demand of good 1 depends only on p_1 and the demand of good 2 depends only on p_2.

Solution: The utility maximization problem is

$$\begin{array}{ll} \text{maximize} & u(x_0, x_1, x_2) = x_0 + \phi_1(x_1) + \phi_2(x_2) \\ \text{subject to} & x_0 + p_1 x_1 + p_2 x_2 \leq w. \end{array}$$

Since the utility function is increasing, the budget constraint binds. Therefore we may assume $x_0 + p_1 x_1 + p_2 x_2 = w$. Substituting for x_0, the problem is equivalent to

$$\text{maximize} \quad w - p_1 x_1 - p_2 x_2 + \phi_1(x_1) + \phi_2(x_2).$$

But since the objective function is the sum of two functions with different variables, this problem is further equivalent to

$$\text{maximize} \quad \phi_l(x_l) - p_l x_l,$$

where $l = 1, 2$. Therefore the solution depends only on the price of that good.

(This problem justifies the usual demand curve-supply curve in introductory (intermediate?) microeconomics: if the utility function is quasi-linear and the nonlinear part is additively separable, you can consider each market separately.)

5. Consider an economy with two periods denoted by $t = 0, 1$ (today and tomorrow). There is a single good at each period. As explained in the lecture, goods are distinguished by time, so introducing time does not change the model (at least formally). The good is perishable, like raw fish, so the good today cannot be transformed into the good tomorrow.

   (a) (5 points) Let p_0, p_1 be the price of 1 unit of good at each period, so actually p_1 is the *future price* (the amount of money you have to pay today in order to be delivered 1 unit of good tomorrow). Let the interest rate be $100r\%$. By arguing how much good you will get tomorrow by reducing the consumption of 1 unit of good today, express $1 + r$ as a function of p_0 and p_1.

   Solution: Suppose you save 1 unit of good today. By the definition of the interest rate, you will get $1 + r$ units of good back tomorrow. On the other hand, if you cut consumption today by 1, you will save p_0 of money (unit of account). Using this fund, you can buy p_0/p_1 units of tomorrow's good, since the price per unit is p_1. Therefore the interest rate is

   $$1 + r = \frac{p_0}{p_1}.$$

   (b) (5 points) Suppose that there are two agents, A and B, and the endowment of each agent is $e^A = e^B = (e_0, e_1)$. (So A and B go fishing and fish e_0 at $t = 0$ and e_1 at $t = 1$.) Each agent has the utility function

   $$u^A(x_0, x_1) = (1 - \beta^A) \log x_0 + \beta^A \log x_1,$$
   $$u^B(x_0, x_1) = (1 - \beta^B) \log x_0 + \beta^B \log x_1.$$

   Let the price be $p_0 = 1$ and $p_1 = p$. Compute the demand of agent A, given the price p.

Solution: The utility function is Cobb-Douglas. Therefore the demand is

$$(x_0^A, x_1^A) = \left((1-\beta^A)w, \frac{\beta^A w}{p}\right),$$

where $w = e_0 + pe_1$.

(c) (5 points) Compute the equilibrium price and interest rate.

Solution: The aggregate demand of good 0 is

$$x_0^A + x_0^B = (1-\beta^A)(e_0+pe_1) + (1-\beta^B)(e_0+pe_1) = (2-\beta^A-\beta^B)(e_0+pe_1).$$

The aggregate supply of good 0 is $e_0 + e_0 = 2e_0$. Therefore the market clearing condition is

$$(2-\beta^A-\beta^B)(e_0+pe_1) = 2e_0 \iff p = \frac{\beta^A+\beta^B}{2-\beta^A+\beta^B}\frac{e_0}{e_1}.$$

The interest rate is then

$$1+r = \frac{1}{p} = \left(\frac{2}{\beta^A+\beta^B} - 1\right)\frac{e_1}{e_0}.$$

(d) (5 points) Assume $\beta^A > \beta^B$. Which agent is saving? Which agent is borrowing?

Solution: Since $\beta^A > \beta^B$, by the formula for the demand, we have $x_0^A < e_0 < x_0^B$. Therefore A saves and lends to B (A sells today's good, buys a future contract, and receives a delivery tomorrow) and B finances his time 0 consumption by borrowing and pays back to A at $t = 1$ (B sells a future contract, buys today's good, and deliver tomorrow's good to A).