Mathematical Economics
Midterm Exam 2

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Name: ______________________________

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- The exam time is 80 minutes.
- This exam has 4 questions on 6 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Submit your entire exam sheet before leaving the room, even if some parts are empty or you intend to drop the class.

Question:	1	2	3	4	Total
Points:	25	25	25	25	100
Score:					

1. Consider an economy with I agents with utility functions (u_i). An *allocation* is denoted by (x_i), where x_i is the consumption bundle of agent i.

(a) (5 points) What does it mean that an allocation (y_i) Pareto dominates the allocation (x_i)? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

> **Solution:** (y_i) Pareto dominates (x_i) if
>
> 1. $u_i(y_i) \geq u_i(x_i)$ for all i, and
> 2. $u_i(y_i) > u_i(x_i)$ for some i.
>
> That is, an allocation Pareto dominates another if the everybody weakly prefers the former and somebody strictly prefers the former.

(b) (5 points) Let the initial endowment be (e_i). What does it mean that the allocation (x_i) is Pareto efficient? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

> **Solution:** (x_i) is Pareto efficient if
>
> 1. (x_i) is feasible, so $\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i$, and
> 2. there are no other feasible allocation (y_i) that Pareto dominate (x_i).
>
> That is, an allocation is Pareto efficient if it is impossible to make somebody better off without making somebody worse off.

(c) (5 points) What does the first welfare theorem say? You can explain in words.

> **Solution:** The first welfare theorem says that the competitive equilibrium allocation is Pareto efficient, provided that utility functions are locally non-satiated. (1 point for referring to the equilibrium allocation, 3 points for saying Pareto efficient, and 1 point for mentioning local non-satiation.)

(d) (10 points) How are the assumptions of the first and second welfare theorems different? You can explain in words.

> **Solution:** For the first welfare theorem, the only assumption is local non-satiation (2 points). For the second welfare theorem, we assume that utility functions are continuous, quasi-concave, locally non-satiated, and that the allocation (x_i) is interior—that is, $x_i \gg 0$ for all i (2 points for each of these four assumptions).

2. Consider an economy with two agent with Cobb-Douglas utility functions

$$u_1(x_1, x_2) = \frac{2}{3}\log x_1 + \frac{1}{3}\log x_2,$$
$$u_2(x_1, x_2) = \frac{1}{3}\log x_1 + \frac{2}{3}\log x_2.$$

Assume that the endowments are $e_1 = (1, 2)$ and $e_2 = (1, 3)$.

(a) (5 points) Consider the allocation (x_1, x_2) (for agent 1) and $(2 - x_1, 5 - x_2)$ (for agent 2). Compute the marginal rate of substitution

$$\frac{\partial u_i}{\partial x_2} \Big/ \frac{\partial u_i}{\partial x_1}$$

for each agent $i = 1, 2$ at this allocation.

> **Solution:** For Cobb-Douglas utility $u(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2$, we have
> $$\frac{\partial u}{\partial x_2} \Big/ \frac{\partial u}{\partial x_1} = \frac{(1-\alpha)x_1}{\alpha x_2}.$$
> Therefore the MRS of agent 1 is $\frac{x_1}{2x_2}$ and the MRS of agent 2 is $\frac{2(2-x_1)}{5-x_2}$.

(b) (5 points) Show that the initial endowment $\{(1, 2), (1, 3)\}$ is Pareto inefficient.

> **Solution:** Substituting $(x_1, x_2) = (1, 2)$, we have $\text{MRS}_1 = \frac{1}{4}$ and $\text{MRS}_2 = \frac{2}{3}$, which are not equal. Therefore the allocation is Pareto inefficient.

(c) (5 points) Show that the allocation $\{(1, 1), (1, 4)\}$ is Pareto efficient.

> **Solution:** Substituting $(x_1, x_2) = (1, 1)$, we have $\text{MRS}_1 = \frac{1}{2}$ and $\text{MRS}_2 = \frac{1}{2}$, which are equal. Since the given allocation exhausts the resources, it is Pareto efficient.

(d) (10 points) Compute the price vector $p = (p_1, p_2)$ (with $p_1 = 1$) and transfer payments (t_1, t_2) such that the price p and the allocation $\{(1, 1), (1, 4)\}$ constitute a competitive equilibrium with transfer payments.

> **Solution:** Using the Lagrangian, we know that in equilibrium we have $\frac{\partial u}{\partial x_l} = \lambda p_l$, so taking the ratio, MRS is equal to the price ratio. Since $p_1 = 1$ and $\text{MRS} = 1/2$, it must be $p_2 = 1/2$. To make the bundle $(1, 1)$ just affordable, agent 1 must give up $t_1 = 1/2 \times 1 = 1/2$ of unit of account. Therefore $t_1 = 1/2$ and $t_2 = -1/2$. (5 points for the price, 5 points for the transfer payments.)

3. Consider an economy with two countries, $i = A, B$, and three consumption goods, $l = 1, 2, 3$. Both countries have labor endowment $e_1 = e_2 = 1$. The utility functions are

$$u_A(x_1, x_2, x_3) = \frac{1}{2}\log x_1 + \frac{1}{4}\log x_2 + \frac{1}{4}\log x_3,$$
$$u_B(x_1, x_2, x_3) = \frac{1}{3}\log x_1 + \frac{1}{3}\log x_2 + \frac{1}{3}\log x_3.$$

Each country can produce the consumption goods from labor using the linear technology $y = a_{il}e$, where e is labor input, y is output of good l, and $a_{il} > 0$ is the productivity. Assume that productivities are

$$(a_{A1}, a_{A2}, a_{A3}) = (4, 2, 2),$$
$$(a_{B1}, a_{B2}, a_{B3}) = (1, 1, 2).$$

(a) (5 points) What is the definition of comparative advantage of country A over B? Compute the comparative advantage for each industry.

Solution: The comparative advantage is the ratio of productivities a_{Al}/a_{Bl} (3 points). Therefore they are

$$\left(\frac{a_{A1}}{a_{B1}}, \frac{a_{A2}}{a_{B2}}, \frac{a_{A3}}{a_{B3}}\right) = (4, 2, 1).$$

(2 points for numerical values.)

(b) (5 points) Given the price $p = (p_1, p_2, p_3)$ and the wage w_A of country A, compute the demand of country A.

Solution: The income of country A is $w_A \times e_1 = w_A$. Since the utility function is Cobb-Douglas, the demand is

$$(x_{A1}, x_{A2}, x_{A3}) = \left(\frac{w_A}{2p_1}, \frac{w_A}{4p_2}, \frac{w_A}{4p_3}\right).$$

(c) (5 points) Assuming that both countries produce good 2 in free trade and setting $p_2 = 1$, compute p_1, p_3, w_A, w_B.

Solution: If country i produces good l, by profit maximization (zero profit) it must be $p_l a_{il} = w_i$. Setting $p_2 = 1$, we get $w_A = a_{A2} = 2$, $w_B = a_{B2} = 1$, $p_1 = w_A/a_{A1} = 2/4 = 1/2$, and $p_3 = w_B/a_{B3} = 1/2$. (2 points for one, 1 point each for the rest.)

(d) (5 points) Compute the free trade equilibrium consumption in each country.

Solution: Substituting the prices into the above formula, we get

$$(x_{A1}, x_{A2}, x_{A3}) = \left(\frac{2}{2(1/2)}, \frac{2}{4}, \frac{2}{4(1/2)}\right) = (2, 1/2, 1).$$

Similarly, the consumption of country B is

$$(x_{B1}, x_{B2}, x_{B3}) = \left(\frac{1}{3(1/2)}, \frac{1}{3}, \frac{1}{3(1/2)}\right) = (2/3, 1/3, 2/3).$$

(e) (5 points) Compute the labor allocation across each industry for each country.

Solution: In order to produce

$$x_{A1} + x_{B1} = 2 + \frac{2}{3} = \frac{8}{3}$$

units of good 1, country A has to hire labor

$$e_{A1} = \frac{8}{3}/4 = \frac{2}{3}.$$

Therefore $e_{A2} = \frac{1}{3}$ and $e_{A3} = 0$.
In order to produce

$$x_{A3} + x_{B3} = 1 + \frac{2}{3} = \frac{5}{3}$$

units of good 3, country B has to hire labor

$$e_{B3} = \frac{5}{3}/2 = \frac{5}{6}.$$

Therefore $e_{B2} = \frac{1}{6}$ and $e_{B1} = 0$.
Then the world output of good 2 is

$$2 \times \frac{1}{3} + 1 \times \frac{1}{6} = \frac{5}{6} = x_{A2} + x_{B2},$$

so the markets clear.

4. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility function of each agent is

$$u(x_1, x_2) = x_1 x_2.$$

Suppose that the initial endowments are $e_1 = (4, 4)$, $e_2 = (20, 8)$, and $e_3 = (4, 16)$. In answering questions below, in order to make the notation consistent use x_{il} for consumption of good l by agent i. (So x_{12} is consumption of good 2 by agent 1, for example.) Also, use $p_1 = 1$ and $p_2 = p$ for the prices.

(a) (5 points) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

> **Solution:** Using the Cobb-Douglas formula, the demand of agents 1, 2 are
>
> $$(x_{11}, x_{12}) = \left(\frac{4+4p}{2}, \frac{4+4p}{2p}\right),$$
> $$(x_{21}, x_{22}) = \left(\frac{20+8p}{2}, \frac{20+8p}{2p}\right).$$
>
> Therefore by market clearing we have
>
> $$(2+2p) + (10+4p) = 4 + 20 \iff p = 2.$$
>
> The demand is $(x_{11}, x_{12}) = (6, 3)$, $(x_{21}, x_{22}) = (18, 9)$. The utility level is $u_1 = 18$ and $u_2 = 162$.

(b) (7 points) Compute the free trade equilibrium price and allocation.

> **Solution:** Using the Cobb-Douglas formula, the demand of agent 3 is
>
> $$(x_{31}, x_{32}) = \left(\frac{4+16p}{2}, \frac{4+16p}{2p}\right).$$
>
> Hence by market clearing, we have
>
> $$(2+2p) + (10+4p) + (2+8p) = 4 + 20 + 4 \iff p = 1.$$
>
> Again using the Cobb-Douglas formula, the consumption of each agents are $(x_{11}, x_{12}) = (4, 4)$, $(x_{21}, x_{22}) = (14, 14)$, and $(x_{31}, x_{32}) = (10, 10)$.

(c) (3 points) Compute the utility level of each agent and determine who gained from trade and who lost.

> **Solution:** Utility levels are $u_1 = 16 < 18$, $u_2 = 196 > 162$, and $u_3 = 100 > 64 = 4 \times 16$. Therefore agents 2 and 3 gained from trade and agent 1 lost.

(d) (10 points) Find a tax scheme in country A such that free trade is Pareto improving. Compute the free trade equilibrium after transfer payments and verify that every agent is better off.

> **Solution:** Since every agent has the same utility function and demand is linear in wealth (since it is Cobb-Douglas), reallocating wealth does not

change the aggregate demand. Therefore no matter what tax we impose, the equilibrium price will still be $p = 1$. In order to make the autarky allocation just affordable, agent 1 should transfer

$$(6 + 3) = (4 + 4) - t_1 \iff t_1 = -1,$$

and consequently $t_2 = -t_1 = 1$. After the transfer, the demand of agent 1 is $(9/2, 9/2)$ and agent 2 is $(27/2, 27/2)$. The utility of agent 1 is $(9/2)^2 = 81/4 > 18$ and agent 2 is $(27/2)^2 = 729/4 > 162$. Therefore everybody is better off.