Econ 113 Mathematical Economics
Final Exam

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December 7, 2015

Name: _____

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base e $= 2.718281828\ldots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 80 minutes.
- This exam has 8 questions on 9 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	15	10	7	16	7	20	15	100
Score:									

1. (a) (2 points) What was the most interesting topic in this course? (Any nonempty answer gets full credit.)

   (b) (3 points) What is the definition of the Sharpe ratio of a portfolio?

   (c) (5 points) Suppose that the market return is 8%, the market volatility is 18%, and the risk-free rate is 2%. If a hedge fund returns 11% with volatility 29%, is the hedge fund manager skillful or not? Answer based on the capital asset pricing model.

2. Consider an agent with utility function

$$u(x_1, x_2) = \frac{3}{2}x_1^{\frac{2}{3}} + \frac{3}{2}x_2^{\frac{2}{3}}.$$

   Assume that the endowment of the agent is (e_1, e_2) and the price of each good is $p_1 = 1$ and $p_2 = p > 0$.

   (a) (2 points) Compute the wealth w of the agent.

   (b) (2 points) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$. You may use w.

   (c) (2 points) Derive the first-order conditions with respect to x_1 and x_2.

(d) (3 points) Express the demand x_1, x_2 in terms of λ and p.

(e) (3 points) Solve for the demand (x_1, x_2) using only p, e_1, e_2.

(f) (3 points) Prove that x_2 is decreasing in p, so the demand is downward sloping.

3. Consider an economy with two agents and two goods. The utility functions are

$$u_1(x_1, x_2) = \min\{x_1, x_2\},$$
$$u_2(x_1, x_2) = \min\{x_1, x_2, 4\}.$$

The initial endowments are $e_1 = (3, 7)$ and $e_2 = (7, 3)$.

(a) (3 points) Show that the price vector $(p_1, p_2) = (1, 1)$ and the allocation $x_1 = x_2 = (5, 5)$ constitute a competitive equilibrium.

(b) (3 points) Show that the equilibrium allocation is Pareto inefficient.

(c) (4 points) Does this example contradict the first welfare theorem? Answer yes or no, and explain why.

4. (7 points) Explain why free trade is a good idea. If a government imposes some trade policy and wishes to adopt free trade, how is it possible to make a Pareto improvement?

5. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility functions are

$$u_1(x_1, x_2) = x_1^2 x_2,$$
$$u_2(x_1, x_2) = x_1 x_2^2,$$
$$u_3(x_1, x_2) = x_1 x_2.$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (18, 6)$. In answering questions below, in order to make the notation consistent use x_{il} for consumption of good l by agent i. (So x_{12} is consumption of good 2 by agent 1, for example.) Also, use $p_1 = 1$ and $p_2 = p$ for the prices.

(a) (4 points) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

(b) (4 points) Compute the free trade equilibrium price and allocation.

(c) (3 points) Compute the utility level of each agent and determine who gained from trade and who lost.

(d) (5 points) Find a tax scheme in country A such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

6. (7 points) According to the capital asset pricing model, in which assets should you invest? Explain.

7. Consider an economy with one physical good (raw fish), two periods ($t = 0, 1$), and

two states at $t = 1$, denoted by $s = 1, 2$. Let π_s be the probability of state $s = 1, 2$, where $\pi_1 + \pi_2 = 1$. Suppose that agents (fishermen) have utility functions

$$u(x_0, x_1, x_2) = (1 - \beta) \log x_0 + \pi_1 \beta \log x_1 + \pi_2 \beta \log x_2,$$

where $0 < \beta < 1$ is a parameter and x_0, x_1, x_2 are consumption of fish at $t = 0$, state 1, and state 2.

(a) (2 points) How many goods are there in this economy?

(b) (2 points) Suppose that fishermen are sophisticated and trade future contracts that are contingent on states. For example, a state 1 contract delivers 1 fish in state 1. Let $p_0 = 1$ be the price of fish at $t = 0$ and p_s ($s = 1, 2$) be the future price state s future contract. If an agent has wealth w, what is his demand?

(c) (5 points) Let e_0, e_1, e_2 be the aggregate endowment of fish and $W = e_0 + p_1 e_1 + p_2 e_2$ be the aggregate wealth. Show that

$$\frac{1 - \beta}{e_0} = \frac{\pi_1 \beta}{p_1 e_1} = \frac{\pi_2 \beta}{p_2 e_2} = \frac{1}{W}.$$

(d) (3 points) Compute the future prices p_1, p_2.

(e) (4 points) Consider an asset that pays out the aggregate endowment of fish at $t = 1$ as dividend. Compute the asset price.

(f) (4 points) Compute the gross risk-free rate in this economy.

8. Suppose that there are two assets, a stock and a bond. The current stock price is 100 and can either go up to 120 or go down to 90 tomorrow. The current bond price is 90 and pays 100 for sure tomorrow. In answering the questions below, always use fractions.

   (a) (5 points) Let u, d stand for the up and down states. Let p_u, p_d be the state prices. (That is, p_u is the price of an asset that pays 1 in state u and 0 in state d. Similarly for p_d.) Derive two equations that p_u, p_d satisfy.

   (b) (4 points) Compute p_u, p_d.

(c) (2 points) Compute the price of a call option with strike 100. (A call option is the right to buy the stock at a specified strike price.)

(d) (2 points) Compute the price of a put option with strike 100. (A put option is the right to sell the stock at a specified strike price.)

(e) (2 points) Compute the price of a convertible bond that promises to pay 100 tomorrow. (A convertible bond is a promise to pay 100, with an option to deliver the stock instead.)

You can detach this sheet and use as a scratch paper.