Econ 113 Mathematical Economics
Final Exam

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Name: ______________________________

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2-2x+1 \geq 0$ is true but not obvious. You need to argue $x^2-2x+1=(x-1)^2 \geq 0$.
- The exam time is 80 minutes.
- This exam has 8 questions on 8 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	15	10	7	16	7	20	15	100
Score:									

1. (a) (2 points) What was the most interesting topic in this course? (Any nonempty answer gets full credit.)

   Solution: Write whatever you want.

   (b) (3 points) What is the definition of the Sharpe ratio of a portfolio?

   Solution: If a portfolio has expected returm μ and volatility σ and the risk-free rate is r, then the Sharpe ratio is $\frac{\mu - r}{\sigma}$.

   (c) (5 points) Suppose that the market return is 8%, the market volatility is 18%, and the risk-free rate is 2%. If a hedge fund returns 11% with volatility 29%, is the hedge fund manager skillful or not? Answer based on the capital asset pricing model.

   Solution: The Sharpe ratio of the market is $\frac{8-2}{18} = \frac{1}{3}$. The Sharpe ratio of the hedge fund is $\frac{11-2}{29} = \frac{9}{29} < \frac{1}{3}$. According to CAPM, any optimal portfolio achieves the maximum Sharpe ratio. Therefore the hedge fund manager is *not* skillful.

2. Consider an agent with utility function

$$u(x_1, x_2) = \frac{3}{2} x_1^{\frac{2}{3}} + \frac{3}{2} x_2^{\frac{2}{3}}.$$

Assume that the endowment of the agent is (e_1, e_2) and the price of each good is $p_1 = 1$ and $p_2 = p > 0$.

   (a) (2 points) Compute the wealth w of the agent.

   Solution: $w = e_1 + pe_2$.

   (b) (2 points) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$. You may use w.

   Solution:

$$L(x_1, x_2, \lambda) = \frac{3}{2} x_1^{\frac{2}{3}} + \frac{3}{2} x_2^{\frac{2}{3}} + \lambda(w - x_1 - px_2).$$

   (c) (2 points) Derive the first-order conditions with respect to x_1 and x_2.

   Solution:

$$x_1 : \quad x_1^{-\frac{1}{3}} - \lambda = 0,$$

$$x_2 : \quad x_2^{-\frac{1}{3}} - \lambda p = 0.$$

(d) (3 points) Express the demand x_1, x_2 in terms of λ and p.

Solution: Solving above, we get $x_1 = \lambda^{-3}$ and $x_2 = (\lambda p)^{-3}$.

(e) (3 points) Solve for the demand (x_1, x_2) using only p, e_1, e_2.

Solution: By the budget constraint, we get

$$w = x_1 + px_2 = \lambda^{-3}(1 + pp^{-3}) \iff \lambda^{-3} = \frac{w}{1 + p^{-2}}.$$

Substituting in the previous result, we get

$$(x_1, x_2) = \left(\frac{e_1 + pe_2}{1 + p^{-2}}, \frac{e_1 + pe_2}{p^3 + p} \right).$$

(f) (3 points) Prove that x_2 is decreasing in p, so the demand is downward sloping.

Solution: Since $x_2 = \frac{e_1/p + e_2}{p^2+1}$, the denominator is increasing in p, and the numerator is decreasing in p, x_2 is decreasing in p.

3. Consider an economy with two agents and two goods. The utility functions are

$$u_1(x_1, x_2) = \min\{x_1, x_2\},$$
$$u_2(x_1, x_2) = \min\{x_1, x_2, 4\}.$$

The initial endowments are $e_1 = (3, 7)$ and $e_2 = (7, 3)$.

(a) (3 points) Show that the price vector $(p_1, p_2) = (1, 1)$ and the allocation $x_1 = x_2 = (5, 5)$ constitute a competitive equilibrium.

Solution: Agent 1 has a Leontief utility function, so demand should satisfy $x_{11} = x_{12}$. With price $(1, 1)$, the wealth is $3 + 7 = 10$, so the demand is $(5, 5)$. Agent 2's utility is at most 4, and the bundle $(5, 5)$ gives utility 4 and is affordable. This allocation is feasible. Therefore it constitutes a competitive equilibrium.

(b) (3 points) Show that the equilibrium allocation is Pareto inefficient.

Solution: The allocation $x_1 = (6, 6)$ and $x_2 = (4, 4)$ is feasible and makes agent 1 strictly better off, while making agent 2 indifferent. Therefore it Pareto improves the competitive equilibrium allocation, so the competitive equilibrium is Pareto inefficient.

(c) (4 points) Does this example contradict the first welfare theorem? Answer yes or no, and explain why.

> **Solution:** No (1 point), because agent 2's utility function is not locally non-satiated (3 points).

4. (7 points) Explain why free trade is a good idea. If a government imposes some trade policy and wishes to adopt free trade, how is it possible to make a Pareto improvement?

> **Solution:** Free trade is great because by the first welfare theorem, the equilibrium allocation is Pareto efficient. If a small country has some trade policy, it is possible to make a Pareto improvement by taxing/subsidizing citizens so that the allocation under the current trade policy is just affordable in free trade. This tax/subsidy is budget feasible because the GDP computed using world price must be the same whether we compute from consumption (demand) or endowment (supply).

5. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility functions are

$$\begin{aligned} u_1(x_1, x_2) &= x_1^2 x_2, \\ u_2(x_1, x_2) &= x_1 x_2^2, \\ u_3(x_1, x_2) &= x_1 x_2. \end{aligned}$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (18, 6)$. In answering questions below, in order to make the notation consistent use x_{il} for consumption of good l by agent i. (So x_{12} is consumption of good 2 by agent 1, for example.) Also, use $p_1 = 1$ and $p_2 = p$ for the prices.

(a) (4 points) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

> **Solution:** Using the Cobb-Douglas formula, the demand of agents 1, 2 are
>
> $$\begin{aligned} (x_{11}, x_{12}) &= \left(\frac{2(3+3p)}{3}, \frac{3+3p}{3p} \right) = (2+2p, 1+1/p), \\ (x_{21}, x_{22}) &= \left(\frac{3+3p}{3}, \frac{2(3+3p)}{3p} \right) = (1+p, 2+2/p). \end{aligned}$$
>
> Therefore by market clearing we have
>
> $$(2+2p) + (1+p) = 3+3 \iff p = 1.$$

The demand is $(x_{11}, x_{12}) = (4, 2)$, $(x_{21}, x_{22}) = (2, 4)$. The utility level is $u_1^a = 4^2 \cdot 2 = 32$ and $u_2^a = 2 \cdot 4^2 = 32$.

(b) (4 points) Compute the free trade equilibrium price and allocation.

Solution: Using the Cobb-Douglas formula, the demand of agent 3 is

$$(x_{31}, x_{32}) = \left(\frac{18+6p}{2}, \frac{18+6p}{2p}\right) = (9+3p, 3+9/p).$$

Hence by market clearing, we have

$$(2+2p) + (1+p) + (9+3p) = 3+3+18 \iff p = 2.$$

Again using the Cobb-Douglas formula, the consumption of each agents are $(x_{11}, x_{12}) = (6, 3/2)$, $(x_{21}, x_{22}) = (3, 3)$, and $(x_{31}, x_{32}) = (15, 15/2)$.

(c) (3 points) Compute the utility level of each agent and determine who gained from trade and who lost.

Solution: Utility levels are

$$\begin{aligned} u_1^f &= 6^2 \cdot 3/2 = 54 > 32 = u_1^a, \\ u_2^f &= 3 \cdot 3^2 = 27 < 32 = u_2^a, \\ u_3^f &= 15 \cdot 15/2 = \frac{225}{2} > 108 = 18 \cdot 6 = u_3^a. \end{aligned}$$

Therefore agents 1 and 3 gained from trade and agent 2 lost.

(d) (5 points) Find a tax scheme in country A such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

Solution: Consider a hypothetical economy in which agents start with the autarky allocation $e_1' = (4, 2)$, $e_2' = (2, 4)$, and $e_3' = (18, 6)$. Using the Cobb-Douglas formula, the demand is

$$\begin{aligned} (x_{11}, x_{12}) &= \left(\frac{2(4+2p)}{3}, \frac{4+2p}{3p}\right), \\ (x_{21}, x_{22}) &= \left(\frac{2+4p}{3}, \frac{2(2+4p)}{3p}\right), \\ (x_{31}, x_{32}) &= \left(\frac{18+6p}{2}, \frac{18+6p}{2p}\right). \end{aligned}$$

The market clearing condition is

$$\frac{2(4+2p)}{3} + \frac{2+4p}{3} + \frac{18+6p}{2} = 4+2+18 \iff p = \frac{35}{17}.$$

This price should be the free trade price after the tax and transfer. To make the autarky equilibrium allocation just affordable, taxes should be set such that $p \cdot x_i^a = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i^a)$. Therefore

$$t_1 = (1, 35/17) \cdot (3-4, 3-2) = -1 + \frac{35}{17} = \frac{18}{17},$$
$$t_2 = (1, 35/17) \cdot (3-2, 3-4) = 1 - \frac{35}{17} = -\frac{18}{17}.$$

With these tax/transfers and price, since the autarky allocation is affordable, the equilibrium allocation Pareto dominates the autarky allocation.

6. (7 points) According to the capital asset pricing model, in which assets should you invest? Explain.

Solution: According to the capital asset pricing model, investors hold the same portfolio of risky assets. Since all assets must be held by someone, this risky portfolio is the entire market (market portfolio). The fraction of wealth invested in the market portfolio is the investor's risk tolerance relative to the average risk tolerance in the economy. The remaining wealth should be invested in the risk-free asset.

7. Consider an economy with one physical good (raw fish), two periods ($t = 0, 1$), and two states at $t = 1$, denoted by $s = 1, 2$. Let π_s be the probability of state $s = 1, 2$, where $\pi_1 + \pi_2 = 1$. Suppose that agents (fishermen) have utility functions

$$u(x_0, x_1, x_2) = (1-\beta)\log x_0 + \pi_1 \beta \log x_1 + \pi_2 \beta \log x_2,$$

where $0 < \beta < 1$ is a parameter and x_0, x_1, x_2 are consumption of fish at $t = 0$, state 1, and state 2.

(a) (2 points) How many goods are there in this economy?

Solution: 3, because goods must be distinguished by time and states of the world.

(b) (2 points) Suppose that fishermen are sophisticated and trade future contracts that are contingent on states. For example, a state 1 contract delivers 1 fish in state 1. Let $p_0 = 1$ be the price of fish at $t = 0$ and p_s ($s = 1, 2$) be the future price state s future contract. If an agent has wealth w, what is his demand?

Solution: Since $\pi_1 + \pi_2 = 1$, we have $(1-\beta) + \pi_1\beta + \pi_2\beta = 1$. Since the utility function is Cobb-Douglas, the demand is

$$(x_0, x_1, x_2) = \left((1-\beta)w, \frac{\pi_1\beta w}{p_1}, \frac{\pi_2\beta w}{p_2}\right).$$

(c) (5 points) Let e_0, e_1, e_2 be the aggregate endowment of fish and $W = e_0 + p_1e_1 + p_2e_2$ be the aggregate wealth. Show that

$$\frac{1-\beta}{e_0} = \frac{\pi_1\beta}{p_1e_1} = \frac{\pi_2\beta}{p_2e_2} = \frac{1}{W}.$$

Solution: By the previous question, the demand is linear in individual wealth w. Adding demand across agents, by market clearing we have

$$(e_0, e_1, e_2) = \left((1-\beta)W, \frac{\pi_1\beta W}{p_1}, \frac{\pi_2\beta W}{p_2}\right).$$

Rearranging terms, we obtain the desired equation.

(d) (3 points) Compute the future prices p_1, p_2.

Solution: By the previous result, we get

$$p_s = \frac{\beta}{1-\beta}\pi_s\frac{e_0}{e_s}.$$

(e) (4 points) Consider an asset that pays out the aggregate endowment of fish at $t = 1$ as dividend. Compute the asset price.

Solution: The asset price is

$$q = p_1e_1 + p_2e_2 = \frac{\beta}{1-\beta}\pi_1e_0 + \frac{\beta}{1-\beta}\pi_2e_0 = \frac{\beta}{1-\beta}e_0.$$

(f) (4 points) Compute the gross risk-free rate in this economy.

Solution: Let R_f be the gross risk-free rate. The price of a bond that pays 1 for sure at $t = 1$ is $\frac{1}{R_f} = p_1 + p_2$. Therefore

$$R_f = \frac{1}{p_1 + p_2} = \frac{1-\beta}{\beta e_0}\frac{1}{\frac{\pi_1}{e_1} + \frac{\pi_2}{e_2}}.$$

8. Suppose that there are two assets, a stock and a bond. The current stock price is 100 and can either go up to 120 or go down to 90 tomorrow. The current bond price is 90 and pays 100 for sure tomorrow. In answering the questions below, always use fractions.

(a) (5 points) Let u, d stand for the up and down states. Let p_u, p_d be the state prices. (That is, p_u is the price of an asset that pays 1 in state u and 0 in state d. Similarly for p_d.) Derive two equations that p_u, p_d satisfy.

Solution: Since the stock pays 120 in the up state and 90 in the down state, its price must be $120p_u + 90p_d$. Therefore

$$120p_u + 90p_d = 100.$$

Similarly, accounting the bond price, we obtain $100p_u + 100p_d = 90$.

(b) (4 points) Compute p_u, p_d.

Solution: Solving the above equations, we get $p_u = \frac{19}{30}$ and $p_d = \frac{4}{15}$.

(c) (2 points) Compute the price of a call option with strike 100. (A call option is the right to buy the stock at a specified strike price.)

Solution: The call option pays out $\max\{120 - 100, 0\} = 20$ in the up state and $\max\{90 - 100, 0\} = 0$ in the down state. Therefore its price is

$$20p_u + 0p_d = \frac{38}{3}.$$

(d) (2 points) Compute the price of a put option with strike 100. (A put option is the right to sell the stock at a specified strike price.)

Solution: The put option pays out $\max\{100 - 120, 0\} = 0$ in the up state and $\max\{100 - 90, 0\} = 10$ in the down state. Therefore its price is

$$0p_u + 10p_d = \frac{8}{3}.$$

(e) (2 points) Compute the price of a convertible bond that promises to pay 100 tomorrow. (A convertible bond is a promise to pay 100, with an option to deliver the stock instead.)

Solution: Since $90 < 100 < 120$, the convertible bond is exercised only in the down state, so it pays 100 in the up state and 90 in the down state.

Therefore its price is

$$100p_u + 90p_d = \frac{262}{3}.$$