Econ 113 Mathematical Economics
Midterm Exam 1

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Name: ____

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2-2x+1 \geq 0$ is true but not obvious. You need to argue $x^2-2x+1=(x-1)^2 \geq 0$.
- The exam time is 80 minutes.
- This exam has 5 questions on 6 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	Total
Points:	10	30	20	15	25	100
Score:						

1. (a) (5 points) What is the definition of a set being convex? You can explain in words.

Solution: A set is convex if for any two points within the set, the segment joining these two points is entirely contained in the set. Formally, C is convex if for any $x, y \in C$ and $0 \leq \alpha \leq 1$, we have $(1-\alpha)x + \alpha y \in C$.

(b) (5 points) What is the definition of a utility function being quasi-linear?

Solution: Suppose that there are $L+1$ goods indexed by $l = 0, 1, \ldots, L$. Then $u(x_0, x_1, \ldots, x_L)$ is quasi-linear if

$$u(x_0, x_1, \ldots, x_L) = x_0 + \phi(x_1, \ldots, x_L)$$

for some function ϕ.

2. Consider an agent with utility function

$$u(x_1, x_2) = 4\sqrt{x_1} + 2\sqrt{x_2}.$$

Assume that the endowment of the agent is (e_1, e_2) and the price of each good is $p_1 = 1$ and $p_2 = p > 0$.

(a) (5 points) Compute the wealth w of the agent.

Solution: $w = e_1 + pe_2$.

(b) (5 points) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$. You may use w.

Solution:

$$L(x_1, x_2, \lambda) = 4\sqrt{x_1} + 2\sqrt{x_2} + \lambda(w - x_1 - px_2).$$

(c) (5 points) Derive the first-order condition with respect to x_1 and x_2.

Solution:

$$x_1: \quad \frac{2}{\sqrt{x_1}} - \lambda = 0,$$

$$x_2: \quad \frac{1}{\sqrt{x_2}} - \lambda p = 0.$$

(d) (5 points) Express the demand x_1, x_2 in terms of λ and p.

Solution: Solving above, we get $x_1 = \frac{4}{\lambda^2}$ and $x_2 = \frac{1}{\lambda^2 p^2}$.

(e) (5 points) Solve for the demand (x_1, x_2) using only p, e_1, e_2.

Solution: By the budget constraint, we get

$$w = x_1 + px_2 = \lambda^{-2}(4 + 1/p) \iff \lambda^{-2} = \frac{w}{4 + 1/p}.$$

Substituting in the previous result, we get

$$(x_1, x_2) = \left(\frac{4(e_1 + pe_2)}{4 + 1/p}, \frac{1}{p} \frac{e_1 + pe_2}{4p + 1} \right).$$

(f) (5 points) Prove that x_2 is decreasing in p, so the demand is downward sloping.

Solution: Since $x_2 = \frac{e_1/p + e_2}{4p+1}$, the denominator is increasing in p, and the numerator is decreasing in p, x_2 is decreasing in p.

3. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility function of each agent is

$$u_A(x_1, x_2) = x_1 - \frac{8}{x_2},$$
$$u_B(x_1, x_2) = x_1 + 10 \log x_2.$$

Suppose that the initial endowments are $(e_1^A, e_2^A) = (4, 4)$ and $(e_1^B, e_2^B) = (10, 3)$.

(a) (10 points) Consider the problem of maximizing the sum of utilities,

$$\begin{aligned} \text{maximize} \quad & u_A(x_1^A, x_2^A) + u_B(x_1^B, x_2^B) \\ \text{subject to} \quad & x_1^A + x_1^B \le e_1^A + e_1^B, \\ & x_2^A + x_2^B \le e_2^A + e_2^B. \end{aligned}$$

Let $(x_1^A, x_2^A, x_1^B, x_2^B)$ be a solution. Compute x_2^A and x_2^B.

Solution: Since the utility functions are quasi-linear, it suffices to maximize the sum of the nonlinear part. For notational simplicity, let $x_2^A = y$ and $x_2^B = z$. Then the problem becomes

$$\begin{aligned} \text{maximize} \quad & -\frac{8}{y} + 10 \log z \\ \text{subject to} \quad & y + z \le 4 + 3 = 7. \end{aligned}$$

The Lagrangian is

$$L(y, z, \lambda) = -\frac{8}{y} + 10 \log z + \lambda(7 - y - z).$$

The first-order conditions are

$$0 = \frac{\partial L}{\partial y} = \frac{8}{y^2} - \lambda = 0 \iff y = \sqrt{\frac{8}{\lambda}},$$
$$0 = \frac{\partial L}{\partial z} = \frac{10}{z} - \lambda = 0 \iff z = \frac{10}{\lambda}.$$

Substituting into the feasibility constraint, we get

$$y + z = 7 \iff \sqrt{\frac{8}{\lambda}} + \frac{10}{\lambda} = 7.$$

Letting $t = \sqrt{\frac{2}{\lambda}}$, this equation becomes

$$2t + 5t^2 = 7 \iff 5t^2 + 2t - 7 = 0 \iff (5t + 7)(t - 1) = 0 \iff t = 1$$

since $t > 0$. Hence $\lambda = \frac{2}{t^2} = 2$. Therefore $x_2^A = y = \sqrt{8/\lambda} = 2$ and $x_2^B = z = 10/\lambda = 5$.

(b) (10 points) Let the price be $p_1 = 1$ and $p_2 = p$. Compute the equilibrium price p and the allocation.

Solution: In a quasi-linear economy, we know from the lecture notes that the price is equal to the Lagrange multiplier. Therefore $p = \lambda = 2$, and the consumption of good 2 is the same as above. By the budget constraint, the consumption of good 1 is

$$x_1^A = e_1^A + p(e_2^A - x_2^A) = 4 + 2(4 - 2) = 8,$$
$$x_1^B = e_1^B + p(e_2^B - x_2^B) = 10 + 2(3 - 5) = 6.$$

4. Consider an economy with I agents and L goods. Let $u_i(x)$ be the utility function of agent i, assumed to be locally nonsatiated. Let $p = (p_1, \dots, p_L)$ be the price vector.

(a) (5 points) What is the definition of local nonsatiation? You can explain in words.

Solution: A utility function is locally nonsatiated if for any consumption bundle x, we can find another consumption bundle y that is arbitrarily close to x and gives higher utility than x. Formally, u is locally nonsatiated if for any x and $\epsilon > 0$, there exists y such that $\|y - x\| < \epsilon$ and $u(y) > u(x)$.

(b) (10 points) When solving for the equilibrium of an economy with locally nonsatiated utility functions, if $L - 1$ markets clear, the other market automatically clears. Explain in detail why this is true.

Solution: When utility functions are locally nonsatiated, agents spend all their income. This is because if an agent does not spend all his income, by local nonsatiation we can find a consumption bundle that is affordable and gives higher utility, which is a contradiction. By adding individual budget constraints (which hold with equalities), it follows that the value of excess demand (aggregate demand minus supply) is zero. Therefore the L market clearing conditions are not independent, and one is redundant. This is why if $L - 1$ markets clear, the other market automatically clears.

5. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility functions are

$$u_A(x_1, x_2) = 3 \log x_1 + \log x_2,$$
$$u_B(x_1, x_2) = \min \{x_1, x_2\}.$$

The endowments are $(e_1^A, e_2^A) = (8, 0)$ and $(e_1^B, e_2^B) = (4, 7)$.

(a) (2 points) What is the name of the type of agent A's utility function?

Solution: Cobb-Douglas.

(b) (3 points) What is the name of the type of agent B's utility function?

Solution: Leontief.

(c) (5 points) Let the prices be $p_1 = 1$ and $p_2 = p$. Compute the demand of agent A.

Solution: Using the Cobb-Douglas formula, we get

$$(x_1^A, x_2^A) = \left(\frac{3}{4}\frac{8}{1}, \frac{1}{4}\frac{8}{p}\right) = \left(6, \frac{2}{p}\right).$$

(d) (5 points) Compute the demand (x_1^B, x_2^B) of agent B, using p.

Solution: Since B has a Leontief utility, we have $x_1^B = x_2^B$. Since his income is $e_1^B + pe_2^B = 4 + 7p$, the demand is

$$(x_1^B, x_2^B) = \left(\frac{4+7p}{1+p}, \frac{4+7p}{1+p}\right).$$

(e) (5 points) Show that x_2^B is increasing in p.

Solution: Since

$$x_2^B = \frac{4+7p}{1+p} = 7 - \frac{3}{1+p},$$

x_2^B is increasing in p. (If the demand of a good increases as the price increases, it is called a Giffen good. Hence good 2 is a Giffen good for agent B.)

(f) (5 points) Compute the competitive equilibrium.

Solution: By market clearing for good 1, we have

$$6 + \frac{4+7p}{1+p} = 8 + 4 \iff 4 + 7p = 6(1+p) \iff p = 2.$$

Substituting the price $p = 2$ into agent A, B's demand, we obtain

$$(x_1^A, x_2^A) = (6, 1),$$
$$(x_1^B, x_2^B) = (6, 6).$$

You can detach this sheet and use as a scratch paper.