Econ 113 Mathematical Economics
Midterm Exam 2

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Name: ______________________________

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 80 minutes.
- This exam has 4 questions on 6 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	Total
Points:	25	25	25	25	100
Score:					

1. Consider an economy with I agents with utility functions (u_i). An *allocation* is denoted by (x_i), where x_i is the consumption bundle of agent i.

   (a) (5 points) What does it mean that an allocation (y_i) Pareto dominates the allocation (x_i)? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

   > **Solution:** (y_i) Pareto dominates (x_i) if
   >
   > 1. $u_i(y_i) \geq u_i(x_i)$ for all i, and
   > 2. $u_i(y_i) > u_i(x_i)$ for some i.
   >
   > That is, an allocation Pareto dominates another if the everybody weakly prefers the former and somebody strictly prefers the former.

   (b) (5 points) Let the initial endowment be (e_i). What does it mean that the allocation (x_i) is Pareto efficient? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

   > **Solution:** (x_i) is Pareto efficient if
   >
   > 1. (x_i) is feasible, so $\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i$, and
   > 2. there are no other feasible allocation (y_i) that Pareto dominate (x_i).
   >
   > That is, an allocation is Pareto efficient if it is impossible to make somebody better off without making somebody worse off.

   (c) (5 points) What does the first welfare theorem say? You can explain in words.

   > **Solution:** The first welfare theorem says that the competitive equilibrium allocation is Pareto efficient, provided that utility functions are locally non-satiated. (1 point for referring to the equilibrium allocation, 3 points for saying Pareto efficient, and 1 point for mentioning local non-satiation.)

   (d) (10 points) How are the assumptions of the first and second welfare theorems different? You can explain in words.

   > **Solution:** For the first welfare theorem, the only assumption is local non-satiation (2 points). For the second welfare theorem, we assume that utility functions are continuous, quasi-concave, locally non-satiated, and that the allocation (x_i) is interior—that is, $x_i \gg 0$ for all i (2 points for each of these four assumptions).

2. Consider an economy with two countries, $i = A, B$, and three consumption goods, $l = 1, 2, 3$. Both countries have labor endowment $e_A = e_B = 6$. The utility functions

are

$$u_A(x_1,x_2,x_3) = \frac{1}{2}\log x_1 + \frac{1}{4}\log x_2 + \frac{1}{4}\log x_3,$$
$$u_B(x_1,x_2,x_3) = \frac{1}{3}\log x_1 + \frac{1}{3}\log x_2 + \frac{1}{3}\log x_3.$$

Each country can produce the consumption goods from labor using the linear technology $y = a_{il}e$, where e is labor input, y is output of good l, and $a_{il} > 0$ is the productivity. Assume that productivities are

$$(a_{A1}, a_{A2}, a_{A3}) = (4, 2, 2),$$
$$(a_{B1}, a_{B2}, a_{B3}) = (1, 1, 2).$$

(a) (5 points) What is the definition of comparative advantage of country A over B? Compute the comparative advantage for each industry.

Solution: The comparative advantage is the ratio of productivities a_{Al}/a_{Bl} (2 points). Therefore they are

$$\left(\frac{a_{A1}}{a_{B1}}, \frac{a_{A2}}{a_{B2}}, \frac{a_{A3}}{a_{B3}}\right) = (4, 2, 1).$$

(2 points for numerical values.)

(b) (5 points) Given the price $p = (p_1, p_2, p_3)$ and the wage w_A of country A, compute the demand of country A.

Solution: The income of country A is $w_A \times e_A = 6w_A$. Since the utility function is Cobb-Douglas, the demand is

$$(x_{A1}, x_{A2}, x_{A3}) = \left(\frac{6w_A}{2p_1}, \frac{6w_A}{4p_2}, \frac{6w_A}{4p_3}\right).$$

(c) (5 points) Assuming that both countries produce good 2 in free trade and setting $p_2 = 1$, compute p_1, p_3, w_A, w_B.

Solution: If country i produces good l, by profit maximization (zero profit) it must be $p_l a_{il} = w_i$. Setting $p_2 = 1$, we get $w_A = a_{A2} = 2$, $w_B = a_{B2} = 1$, $p_1 = w_A/a_{A1} = 2/4 = 1/2$, and $p_3 = w_B/a_{B3} = 1/2$. (1 point for each of p_1, p_3, w_A, w_B.)

(d) (5 points) Compute the free trade equilibrium consumption in each country.

Solution: Substituting the prices into the above formula, we get

$$(x_{A1}, x_{A2}, x_{A3}) = \left(\frac{6 \times 2}{2(1/2)}, \frac{6 \times 2}{4}, \frac{6 \times 2}{4(1/2)}\right) = (12, 3, 6).$$

Similarly, the consumption of country B is

$$(x_{B1}, x_{B2}, x_{B3}) = \left(\frac{6}{3(1/2)}, \frac{6}{3}, \frac{6}{3(1/2)}\right) = (4, 2, 4).$$

(e) (5 points) Compute the labor allocation across each industry for each country.

Solution: In order to produce

$$x_{A1} + x_{B1} = 12 + 4 = 16$$

units of good 1, country A has to hire labor

$$e_{A1} = 16/4 = 4.$$

Therefore $e_{A2} = 2$ and $e_{A3} = 0$.
In order to produce

$$x_{A3} + x_{B3} = 6 + 4 = 10$$

units of good 3, country B has to hire labor

$$e_{B3} = 10/2 = 5.$$

Therefore $e_{B2} = 1$ and $e_{B1} = 0$.
Then the world output of good 2 is

$$2 \times 2 + 1 \times 1 = 5 = x_{A2} + x_{B2},$$

so the markets clear.

3. Consider a world with L goods indexed by $l = 1, \ldots, L$. Let $p = (p_1, \ldots, p_L)$ be the vector of world prices. Suppose that a small country (hence it does not affect world price p) has I citizens indexed by $i = 1, \ldots, I$, and let $u_i(x)$ be the (locally nonsatiated) utility function of agent i and e_i be the initial endowment.

   (a) (10 points) Suppose that the government is adopting some trade policy (tariff, quotas, etc.), and the equilibrium allocation is (x_i). If the government is neither running a trade surplus nor a deficit, show that it must be

$$\sum_{i=1}^{I} p \cdot (e_i - x_i) = 0.$$

Solution: The value of domestic consumption and endowment (GDP) evaluated at the world price are $\sum_{i=1}^{I} p \cdot x_i$ and $\sum_{i=1}^{I} p \cdot e_i$, respectively. Since the government runs a balanced budget, they must be equal. Therefore

$$\sum_{i=1}^{I} p \cdot x_i = \sum_{i=1}^{I} p \cdot e_i \iff \sum_{i=1}^{I} p \cdot (e_i - x_i) = 0.$$

(b) (10 points) Find a trade policy that weakly Pareto improves the initial trade policy. Explain why your policy is weakly Pareto improving.

Solution: Make transfers such that the initial equilibrium allocation is just affordable under free trade, so let

$$p \cdot x_i = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i).$$

By the previous question, we have $\sum_{i=1}^{I} t_i = 0$, so the government budget balances. Since x_i is affordable, the resulting free trade allocation (x_i^f) weakly Pareto dominates the initial equilibrium allocation (x_i).

(c) (5 points) Currently Japan imposes a tariff of 400 Yen per kilogram on rice import (which is essentially the domestic rice price, so even if you import rice for free you will lose money) in order to protect the domestic rice producers. If you are an economist advising the Japanese government, what would you recommend to do, given the result of the previous question? (There is no "right" answer to this question since I am not giving all the details. Try to answer the best you can.)

Solution: First, abolish the tariff completely. To compensate for the income loss of the rice producers, impose an appropriate income tax to all citizens and transfer the tax revenue to the rice producers.

4. Consider an economy with two countries, $i = A, B$, and two physical goods, $l = 1, 2$. The endowment is $e_A = (10, 1)$ and $e_B = (1, 10)$. The utility function is

$$u(x_1, x_2) = \frac{1}{2} \log x_1 + \frac{1}{2} \log x_2$$

for all agents. Suppose that there are transportation costs, and 50% of the exported goods perish by the time they reach the destination.

(a) (5 points) How many kinds of goods are there in the world? Answer the number and explain the reason.

Solution: 4, because there are two physical goods in two locations. (2 points for the answer, 3 point for the reason.)

(b) (5 points) Explain why a model of international trade with transportation costs can be regarded as a standard Arrow-Debreu model.

Solution: With transportation costs, goods must be distinguished by location as well as physical properties. Since transportation is a kind of technology that transforms a good in one country to some other goods in another country (with potentially many inputs such as fuel used for shipping, the labor service of a captain who operates the airplane or boat, etc.), a model of international trade with transportation costs can be interpreted as a standard general equilibrium model with production.

(c) (5 points) Assuming that country A imports good 2, what is its price? (Set the price of good 1 equal to 1.)

Solution: Relabel the goods by $l = 1, 2, 3, 4$, where good 1 is physical good 1 in country A, good 2 is physical good 2 in country A, and so on. Let (p_1, p_2, p_3, p_4) be the price. By symmetry, the equilibrium price must satisfy $p_1 = p_4 = 1$ and $p_2 = p_3$. If country B exports e units of good 4, it becomes $0.5e$ units of good 2. By doing so, the profit is $(0.5p_2 - p_4)e$. By profit maximization, we get

$$(0.5p_2 - p_4)e = 0 \iff p_2 = 2p_4 = 2.$$

(d) (10 points) Compute the free trade equilibrium. Make sure to compute all prices, consumption, and import/exports in each country.

Solution: By the Cobb-Douglas formula, the demand of country A, B is

$$\left(\frac{10p_1 + p_2}{2p_1}, \frac{10p_1 + p_2}{2p_2}, 0, 0\right) = (6, 3, 0, 0),$$
$$\left(0, 0, \frac{p_3 + 10p_4}{2p_3}, \frac{p_3 + 10p_4}{2p_4}\right) = (0, 0, 3, 6).$$

Therefore the net import is $y_A = (-4, 2)$ for country A and $y_B = (2, -4)$ for country B.