Mathematical Economics
Final Exam

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Name: _____

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base e $= 2.718281828\ldots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 80 minutes.
- This exam has 7 questions on 12 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	6	7	Total
Points:	10	10	15	15	15	15	20	100
Score:								

1. (a) (2 points) What was the most interesting topic in this course? (Any nonempty answer gets full credit.)

   (b) (3 points) What is a call option? Explain.

   (c) (2 points) What is the definition of a convex set?

   (d) (3 points) What is the statement of the separating hyperplane theorem? (There are weak and strong versions. Either of them is fine.)

2. Consider an economy with two goods and an agent with utility function

$$u(x_1, x_2) = -\mathrm{e}^{-x_1} - \mathrm{e}^{-x_2}.$$

   Suppose that the agent has initial wealth w. Let the prices be $p_1 = 1$ and $p_2 = p$.

   (a) (3 points) Write down the Lagrangian for the utility maximization problem.

   (b) (3 points) Using the first-order condition, express x_1, x_2 using p and λ.

(c) (4 points) Express the demand using only p and w.

3. Consider an economy with two goods and two agents. The utility functions are

$$U_1(x_1, x_2) = x_1 - \frac{1}{2x_2^2},$$
$$U_2(x_1, x_2) = -\frac{1}{2x_1^2} + x_2.$$

The endowments are $e_1 = e_2 = (e, e)$, where $e > 0$. Assume that agent 1 can consume good 1 in negative amounts, and agent 2 can consume good 2 in negative amounts.

(a) (4 points) Let the prices be $p_1 = 1$ and $p_2 = p$. Compute the demand of agent 1.

(b) (3 points) Let $z_1(p)$ be the aggregate excess demand of good 1. Compute $z_1(p)$.

(c) (2 points) Show that $z_1(1) = 0$ and $z_1(\infty) = \infty$.

(d) (2 points) Compute $z_1'(1)$.

(e) (4 points) Show that this economy has more than one equilibria if $0 < e < \frac{1}{3}$.

4. Consider an economy with two countries, $i = A, B$, and three consumption goods, $l = 1, 2, 3$. Both countries have labor endowment $e_1 = e_2 = 1$. The utility functions are

$$u_A(x_1, x_2, x_3) = \frac{1}{2}\log x_1 + \frac{1}{4}\log x_2 + \frac{1}{4}\log x_3,$$
$$u_B(x_1, x_2, x_3) = \frac{1}{3}\log x_1 + \frac{1}{3}\log x_2 + \frac{1}{3}\log x_3.$$

Each country can produce the consumption goods from labor using the linear technology $y = a_{il}e$, where e is labor input, y is output of good l, and $a_{il} > 0$ is the productivity. Assume that productivities are

$$(a_{A1}, a_{A2}, a_{A3}) = (4, 2, 2),$$
$$(a_{B1}, a_{B2}, a_{B3}) = (1, 1, 2).$$

(a) (3 points) What is the definition of comparative advantage of country A over B? Compute the comparative advantage for each industry.

(b) (3 points) Given the price $p = (p_1, p_2, p_3)$ and the wage w_A of country A, compute the demand of country A.

(c) (3 points) Assuming that both countries produce good 2 in free trade and setting $p_2 = 1$, compute p_1, p_3, w_A, w_B.

(d) (3 points) Compute the free trade equilibrium consumption in each country.

(e) (3 points) Compute the labor allocation across each industry for each country.

5. Consider an economy with two countries, A, B, and two goods, $l = 1, 2$. There are many agents, and the utility function of agent i is $u_i(x_1, x_2)$, which is increasing, quasi-concave, and differentiable. Let the world price of good 2 equal p.

   (a) (5 points) If all countries adopt free trade, what is the marginal rate of substitution between goods 1 and 2 evaluated at the equilibrium allocation?

   (b) (5 points) Suppose the government of country A is concerned about protecting industry 2 and imposes a tariff, so the domestic price of good 2 in country A is $p_2 = p(1 + \tau)$, where $\tau > 0$ is tariff. If country B adopts free trade, prove that no matter how the government of country A transfers the revenue from tariff to its citizens, the equilibrium is Pareto inefficient.

(c) (5 points) Suppose that you are an economist advising the government for trade policy. Propose a policy that achieves Pareto efficiency but at the same time makes everybody at least as well off as autarky.

6. Suppose that there are two assets, a stock and a (risk-free) bond. The current stock price is 100 and can either go up to 120 or go down to 75 tomorrow. The risk-free interest rate is 5%. In answering the questions below, always use fractions.

   (a) (5 points) Let u, d stand for the up and down states and p_u, p_d be the state

prices. Derive two equations that p_u, p_d satisfy.

(b) (4 points) Compute p_u, p_d.

(c) (2 points) Compute the price of a call option with strike 100.

(d) (2 points) Compute the price of a put option with strike 100.

(e) (2 points) Compute the price of a convertible bond that promises to pay 100 tomorrow. (A convertible bond is a promise to pay 100, with an option to deliver the stock instead.)

7. Consider an economy with two periods, denoted by $t = 0, 1$, and three agents, denoted by $i = 1, 2, 3$. There are two states at $t = 1$, denoted by $s = 1, 2$. The two states occur with equal probability $\pi_1 = \pi_2 = 1/2$. Suppose that agent i's utility function is

$$U_i(x_0, x_1, x_2) = u_i(x_0) + \pi_1 u_i(x_1) + \pi_2 u_i(x_2),$$

where x_0, x_1, x_2 denote the consumption at $t = 0$ and states $s = 1, 2$, and the Bernoulli utility functions $u_i(x)$ are given by

$$\begin{aligned} u_1(x) &= \sqrt{2x}, \\ u_2(x) &= \sqrt{2x - 2}, \\ u_3(x) &= \sqrt{2x + 2}. \end{aligned}$$

The initial endowments $e_i = (e_{i0}, e_{i1}, e_{i2})$ are given by

$$\begin{aligned} e_1 &= (1, 2, 5), \\ e_2 &= (2, 2, 4), \\ e_3 &= (3/2, 4, 7/2). \end{aligned}$$

(a) (2 points) What is the name of this type of utility functions?

(b) (2 points) For a given level of consumption, which agent is the most risk averse? Answer based on reasoning.

(c) (6 points) Normalize the price of $t = 0$ good to be $p_0 = 1$. Compute the equilibrium state prices p_1, p_2.

(d) (3 points) Compute the (gross) risk-free interest rate.

(e) (3 points) Consider an asset (stock) that pays out the aggregate endowment as dividend. Compute the ex-dividend stock price (the stock price excluding the dividend) at $t = 0$.

(f) (2 points) Compute the expected stock return at $t = 0$ and show that it is higher than the risk-free rate.

(g) (2 points) Compute the price at $t = 0$ of a call option written on a stock with strike price 10.

You can detach this sheet and use as a scratch paper.