# Mathematical Economics
# Final Exam

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Name: ______________________________

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 80 minutes.
- This exam has 7 questions on 11 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| Points: | 10 | 10 | 15 | 15 | 15 | 15 | 20 | 100 |
| Score: | | | | | | | | |

1. (a) (2 points) What was the most interesting topic in this course? (Any nonempty answer gets full credit.)

   **Solution:** Write whatever you want.

   (b) (3 points) What is a call option? Explain.

   **Solution:** A call option is the right to buy a stock at a specified price (strike price).

   (c) (2 points) What is the definition of a convex set?

   **Solution:** $C$ is convex if for all $x, y \in C$ and $0 \leq \alpha \leq 1$, we have $(1-\alpha)x + \alpha y \in C$.

   (d) (3 points) What is the statement of the separating hyperplane theorem? (There are weak and strong versions. Either of them is fine.)

   **Solution:** If sets $C, D$ are nonempty, convex, and $C \cap D = \emptyset$, then there exists a hyperplane that separates them. More precisely, there exists a nonzero vector $a$ such that

$$a \cdot x \leq a \cdot y$$

   for all $x \in C$ and $y \in D$. (The strong version says that if $C, D$ are nonempty, convex, and $C$ is closed and $D$ is compact, then the above inequality is strict.)

2. Consider an economy with two goods and an agent with utility function

$$u(x_1, x_2) = -\mathrm{e}^{-x_1} - \mathrm{e}^{-x_2}.$$

   Suppose that the agent has initial wealth $w$. Let the prices be $p_1 = 1$ and $p_2 = p$.

   (a) (3 points) Write down the Lagrangian for the utility maximization problem.

   **Solution:**

$$L(x_1, x_2, \lambda) = -\mathrm{e}^{-x_1} - \mathrm{e}^{-x_2} + \lambda(w - x_1 - px_2).$$

   (b) (3 points) Using the first-order condition, express $x_1, x_2$ using $p$ and $\lambda$.

   **Solution:** By FOC, we have $\mathrm{e}^{-x_1} - \lambda = 0 \iff x_1 = -\log \lambda$ and $\mathrm{e}^{-x_2} - \lambda p \iff x_2 = -\log(\lambda p)$.

   (c) (4 points) Express the demand using only $p$ and $w$.

**Solution:** By complementary slackness, we have

$$\begin{aligned} w = x_1 + px_2 &= -\log\lambda - p\log(\lambda p) = -(1+p)\log\lambda - p\log p \\ \iff -\log\lambda &= \frac{w}{1+p} + \frac{p\log p}{1+p}. \end{aligned}$$

Therefore

$$\begin{aligned} x_1 &= \frac{w}{1+p} + \frac{p\log p}{1+p}, \\ x_2 &= \frac{w}{1+p} - \frac{\log p}{1+p}. \end{aligned}$$

3. Consider an economy with two goods and two agents. The utility functions are

$$\begin{aligned} U_1(x_1, x_2) &= x_1 - \frac{1}{2x_2^2}, \\ U_2(x_1, x_2) &= -\frac{1}{2x_1^2} + x_2. \end{aligned}$$

The endowments are $e_1 = e_2 = (e, e)$, where $e > 0$. Assume that agent 1 can consume good 1 in negative amounts, and agent 2 can consume good 2 in negative amounts.

(a) (4 points) Let the prices be $p_1 = 1$ and $p_2 = p$. Compute the demand of agent 1.

**Solution:** Agent 1's budget constraint is $x_1 + px_2 \leq (1+p)e$. Since the utility function is monotonic, we have $x_1 = (1+p)e - px_2$. Therefore agent 1's problem is to maximize

$$(1+p)e - px_2 - \frac{1}{2x_2^2}.$$

The first-order condition is

$$-p + x_2^{-3} = 0 \iff x_2 = p^{-1/3}.$$

Therefore $x_1 = (1+p)e - p^{\frac{2}{3}}$. Thus agent 1's demand is

$$(x_{11}, x_{12}) = \left((1+p)e - p^{\frac{2}{3}}, p^{-\frac{1}{3}}\right).$$

(b) (3 points) Let $z_1(p)$ be the aggregate excess demand of good 1. Compute $z_1(p)$.

**Solution:** Utility functions are symmetric, changing goods 1, 2, and letting $p \to 1/p$, agent 2's demand is

$$(x_{21}, x_{22}) = \left(p^{\frac{1}{3}}, (1 + 1/p)e - p^{-\frac{2}{3}}\right).$$

Hence
$$z_1(p) = x_{11} + x_{21} - 2e = (p-1)e - p^{\frac{2}{3}} + p^{\frac{1}{3}}.$$

(c) (2 points) Show that $z_1(1) = 0$ and $z_1(\infty) = \infty$.

**Solution:** Trivial by direct substitution.

(d) (2 points) Compute $z_1'(1)$.

**Solution:** Since
$$z_1'(p) = e - \frac{2}{3}p^{-1/3} + \frac{1}{3}p^{-2/3},$$
substituting $p = 1$ we obtain
$$z_1'(1) = e - \frac{2}{3} + \frac{1}{3} = e - \frac{1}{3}.$$

(e) (4 points) Show that this economy has more than one equilibria if $0 < e < \frac{1}{3}$.

**Solution:** Since $0 < e < 1/3$, we have $z_1'(1) = e - 1/3 < 0$. Since $z_1(1) = 0$, it follows that $z_1(p) < 0$ when $p > 1$ is sufficiently close to 1. Since $z_1(\infty) = \infty$, by the intermediate value theorem there exists $p^* > 1$ such that $z_1(p^*) = 0$, so there exist multiple equilibria.

4. Consider an economy with two countries, $i = A, B$, and three consumption goods, $l = 1, 2, 3$. Both countries have labor endowment $e_1 = e_2 = 1$. The utility functions are
$$u_A(x_1, x_2, x_3) = \frac{1}{2}\log x_1 + \frac{1}{4}\log x_2 + \frac{1}{4}\log x_3,$$
$$u_B(x_1, x_2, x_3) = \frac{1}{3}\log x_1 + \frac{1}{3}\log x_2 + \frac{1}{3}\log x_3.$$

Each country can produce the consumption goods from labor using the linear technology $y = a_{il}e$, where $e$ is labor input, $y$ is output of good $l$, and $a_{il} > 0$ is the productivity. Assume that productivities are
$$(a_{A1}, a_{A2}, a_{A3}) = (4, 2, 2),$$
$$(a_{B1}, a_{B2}, a_{B3}) = (1, 1, 2).$$

(a) (3 points) What is the definition of comparative advantage of country $A$ over $B$? Compute the comparative advantage for each industry.

**Solution:** The comparative advantage is the ratio of productivities $a_{Al}/a_{Bl}$. Therefore they are
$$\left(\frac{a_{A1}}{a_{B1}}, \frac{a_{A2}}{a_{B2}}, \frac{a_{A3}}{a_{B3}}\right) = (4, 2, 1).$$

(b) (3 points) Given the price $p = (p_1, p_2, p_3)$ and the wage $w_A$ of country $A$, compute the demand of country $A$.

**Solution:** The income of country $A$ is $w_A \times e_1 = w_A$. Since the utility function is Cobb-Douglas, the demand is

$$(x_{A1}, x_{A2}, x_{A3}) = \left( \frac{w_A}{2p_1}, \frac{w_A}{4p_2}, \frac{w_A}{4p_3} \right).$$

(c) (3 points) Assuming that both countries produce good 2 in free trade and setting $p_2 = 1$, compute $p_1, p_3, w_A, w_B$.

**Solution:** If country $i$ produces good $l$, by profit maximization (zero profit) it must be $p_l a_{il} = w_i$. Setting $p_2 = 1$, we get $w_A = a_{A2} = 2$, $w_B = a_{B2} = 1$, $p_1 = w_A / a_{A1} = 2/4 = 1/2$, and $p_3 = w_B / a_{B3} = 1/2$.

(d) (3 points) Compute the free trade equilibrium consumption in each country.

**Solution:** Substituting the prices into the above formula, we get

$$(x_{A1}, x_{A2}, x_{A3}) = \left( \frac{2}{2(1/2)}, \frac{2}{4}, \frac{2}{4(1/2)} \right) = (2, 1/2, 1).$$

Similarly, the consumption of country $B$ is

$$(x_{B1}, x_{B2}, x_{B3}) = \left( \frac{1}{3(1/2)}, \frac{1}{3}, \frac{1}{3(1/2)} \right) = (2/3, 1/3, 2/3).$$

(e) (3 points) Compute the labor allocation across each industry for each country.

**Solution:** In order to produce

$$x_{A1} + x_{B1} = 2 + \frac{2}{3} = \frac{8}{3}$$

units of good 1, country $A$ has to hire labor

$$e_{A1} = \frac{8}{3}/4 = \frac{2}{3}.$$

Therefore $e_{A2} = \frac{1}{3}$ and $e_{A3} = 0$.
In order to produce

$$x_{A3} + x_{B3} = 1 + \frac{2}{3} = \frac{5}{3}$$

units of good 3, country $B$ has to hire labor

$$e_{B3} = \frac{5}{3}/2 = \frac{5}{6}.$$

Therefore $e_{B2} = \frac{1}{6}$ and $e_{B1} = 0$.
Then the world output of good 2 is

$$2 \times \frac{1}{3} + 1 \times \frac{1}{6} = \frac{5}{6} = x_{A2} + x_{B2},$$

so the markets clear.

5. Consider an economy with two countries, $A, B$, and two goods, $l = 1, 2$. There are many agents, and the utility function of agent $i$ is $u_i(x_1, x_2)$, which is increasing, quasi-concave, and differentiable. Let the world price of good 2 equal $p$.

(a) (5 points) If all countries adopt free trade, what is the marginal rate of substitution between goods 1 and 2 evaluated at the equilibrium allocation?

**Solution:** By the first-order condition ($\frac{\partial u_i}{\partial x_l} = \lambda_i p_l$), the marginal rate of substitution at the equilibrium allocation is

$$\frac{\partial u_i}{\partial x_2} / \frac{\partial u_i}{\partial x_1} = p.$$

(b) (5 points) Suppose the government of country $A$ is concerned about protecting industry 2 and imposes a tariff, so the domestic price of good 2 in country $A$ is $p_2 = p(1 + \tau)$, where $\tau > 0$ is tariff. If country $B$ adopts free trade, prove that no matter how the government of country $A$ transfers the revenue from tariff to its citizens, the equilibrium is Pareto inefficient.

**Solution:** Due to the tariff, the marginal rate of substitution of an agent at the equilibrium allocation in contry $A$ is

$$\frac{\partial u_i}{\partial x_2} / \frac{\partial u_i}{\partial x_1} = p(1 + \tau).$$

Since this number is different from the marginal rate of substitution of an agent in country $B$ (which is $p$), the equilibrium is inefficient.

(c) (5 points) Suppose that you are an economist advising the government for trade policy. Propose a policy that achieves Pareto efficiency but at the same time makes everybody at least as well off as autarky.

**Solution:** According to the lecture note, one way to achieve both goals is to adopt a free trade policy but transfer wealth across citizens so that the autarky allocation is just affordable.

6. Suppose that there are two assets, a stock and a (risk-free) bond. The current stock price is 100 and can either go up to 120 or go down to 75 tomorrow. The risk-free interest rate is 5%. In answering the questions below, always use fractions.

(a) (5 points) Let $u, d$ stand for the up and down states and $p_u, p_d$ be the state prices. Derive two equations that $p_u, p_d$ satisfy.

> **Solution:** Since the stock pays 120 in the up state and 90 in the down state, its price must be $120p_u + 90p_d$. Therefore
>
> $$120p_u + 75p_d = 100.$$
>
> Similarly, accounting the bond price, we obtain $105p_u + 105p_d = 100$.

(b) (4 points) Compute $p_u, p_d$.

> **Solution:** Solving the above equations, we get $p_u = \frac{40}{63}$ and $p_d = \frac{20}{63}$.

(c) (2 points) Compute the price of a call option with strike 100.

> **Solution:** The call option pays out $\max\{120 - 100, 0\} = 20$ in the up state and $\max\{75 - 100, 0\} = 0$ in the down state. Therefore its price is
>
> $$20p_u + 0p_d = \frac{800}{63}.$$

(d) (2 points) Compute the price of a put option with strike 100.

> **Solution:** The put option pays out $\max\{100 - 120, 0\} = 0$ in the up state and $\max\{100 - 75, 0\} = 25$ in the down state. Therefore its price is
>
> $$0p_u + 25p_d = \frac{500}{63}.$$

(e) (2 points) Compute the price of a convertible bond that promises to pay 100 tomorrow. (A convertible bond is a promise to pay 100, with an option to deliver the stock instead.)

> **Solution:** Since $75 < 100 < 120$, the convertible bond is exercised only in the down state, so it pays 100 in the up state and 75 in the down state. Therefore its price is
>
> $$100p_u + 75p_d = \frac{5500}{63}.$$

7. Consider an economy with two periods, denoted by $t = 0, 1$, and three agents, denoted by $i = 1, 2, 3$. There are two states at $t = 1$, denoted by $s = 1, 2$. The two states occur with equal probability $\pi_1 = \pi_2 = 1/2$. Suppose that agent $i$'s utility function is

$$U_i(x_0, x_1, x_2) = u_i(x_0) + \pi_1 u_i(x_1) + \pi_2 u_i(x_2),$$

where $x_0, x_1, x_2$ denote the consumption at $t = 0$ and states $s = 1, 2$, and the Bernoulli utility functions $u_i(x)$ are given by

$$\begin{aligned}u_1(x) &= \sqrt{2x},\\ u_2(x) &= \sqrt{2x-2},\\ u_3(x) &= \sqrt{2x+2}.\end{aligned}$$

The initial endowments $e_i = (e_{i0}, e_{i1}, e_{i2})$ are given by

$$\begin{aligned}e_1 &= (1, 2, 5),\\ e_2 &= (2, 2, 4),\\ e_3 &= (3/2, 4, 7/2).\end{aligned}$$

(a) (2 points) What is the name of this type of utility functions?

> **Solution:** Hyperbolic absolute risk aversion (HARA) utilities. Note that a HARA utility has a representation
>
> $$u(x) = \frac{1}{a-1}(ax+b)^{1-\frac{1}{a}}.$$
>
> The given functions correspond to $a = 2$ and $b = 0, -2, 2$.

(b) (2 points) For a given level of consumption, which agent is the most risk averse? Answer based on reasoning.

> **Solution:** The absolute risk aversion (ARA) coefficient is given by
>
> $$-\frac{u''(x)}{u'(x)} = \frac{1}{ax+b}.$$
>
> Since $a > 0$ is common across agents, ARA is larger when $b$ is smaller. Therefore agent 2 is the most risk averse.

(c) (6 points) Normalize the price of $t = 0$ good to be $p_0 = 1$. Compute the equilibrium state prices $p_1, p_2$.

> **Solution:** Since agents have HARA utility with common $a$, by the argument in the lecture note, we can compute the equilibrium price through aggregation. Let $e_s$ be the aggregate endowment in state $s$. Then
>
> $$\begin{aligned}e_0 &= 1 + 2 + \frac{3}{2} = \frac{9}{2},\\ e_1 &= 2 + 2 + 4 = 8,\\ e_2 &= 5 + 4 + \frac{7}{2} = \frac{25}{2}.\end{aligned}$$
>
> Let $\pi_0 = 1$. Then the equilibrium price is given by
>
> $$p_s = C\pi_s(ae_s + b)^{-\frac{1}{a}}$$

for some constant $C > 0$, where $b = \sum_{i=1}^{3} b_i = 0 + (-2) + 2 = 0$. Setting $a = 2$ and $b = 0$, it follows that

$$1 = p_0 = C9^{-\frac{1}{2}} = \frac{1}{3}C \iff C = 3,$$

so

$$p_1 = 3\frac{1}{2}16^{-\frac{1}{2}} = \frac{3}{8},$$
$$p_2 = 3\frac{1}{2}25^{-\frac{1}{2}} = \frac{3}{10}.$$

(d) (3 points) Compute the (gross) risk-free interest rate.

**Solution:** The risk-free asset pays 1 in states $s = 1, 2$. Therefore its price is

$$q = p_1 + p_2 = \frac{3}{8} + \frac{3}{10} = \frac{27}{40}.$$

The gross risk-free rate is

$$R_f = \frac{1}{q} = \frac{40}{27}.$$

(e) (3 points) Consider an asset (stock) that pays out the aggregate endowment as dividend. Compute the ex-dividend stock price (the stock price excluding the dividend) at $t = 0$.

**Solution:** The stock pays $e_s$ in state $s$. Therefore its price is

$$q = p_1 e_1 + p_2 e_2 = \frac{3}{8}8 + \frac{3}{10}\frac{25}{2} = \frac{27}{4}.$$

(f) (2 points) Compute the expected stock return at $t = 0$ and show that it is higher than the risk-free rate.

**Solution:** Since each state occurs with probability 1/2, the expected stock return is

$$\mathrm{E}[R] = \frac{\mathrm{E}[e_s]}{q} = \frac{4}{27}\frac{1}{2}\left(8 + \frac{25}{2}\right) = \frac{41}{27} > \frac{40}{27} = R_f.$$

(g) (2 points) Compute the price at $t = 0$ of a call option written on a stock with strike price 10.

**Solution:** Since the stock price (including the dividend) is $e_s$ in state $s$, and $e_1 = 8 < 10$ and $e_2 = 25/2 > 10$, the call is exercised only in state $s$. Letting

$K = 10$ be the strike, the call price is therefore

$$q = p_2(e_2 - K) = \frac{3}{10}\left(\frac{25}{2} - 10\right) = \frac{3}{4}.$$

You can detach this sheet and use as a scratch paper.