Mathematical Economics
Midterm Exam

Prof. Alexis Akira Toda

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Name: ______________________________

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 80 minutes.
- This exam has 4 questions on 7 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	Total
Points:	25	20	30	25	100
Score:					

1. Consider an economy with L goods and I agents. Agent i has endowment e_i and a locally nonsatiated utility function $u_i(x)$. Let p be the price vector. If the notation bothers you, you may set $L = 2$ and $I = 2$.

   (a) (5 points) What is the definition of local nonsatiation? You may explain in words (maximum 4 points) or mathematically.

   Solution: In words, a utility function is locally nonsatiated if for any bundle you can find an arbitrarily close bundle that gives higher utility. Mathematically, the utility function $u(x)$ is locally nonsatiated if for all x and $\epsilon > 0$, there exists y with $\|y - x\| < \epsilon$ such that $u(y) > u(x)$.

   (b) (5 points) Suppose x_i solves the utility maximization problem

   $$\text{maximize } u_i(x) \text{ subject to } p \cdot x \leq p \cdot e_i.$$

   Explain why it must be the case that $p \cdot x_i = p \cdot e_i$.

   Solution: If $p \cdot x_i < p \cdot e_i$, we can take a small ball with center x_i and radius $\epsilon > 0$ that is contained in the budget set. By local nonsatiation, we can take a bundle y in this ball such that $u_i(y) > u_i(x_i)$, which contradicts utility maximization. See lecture note for a rigorous proof.

   (c) (5 points) What does it mean that an allocation (y_i) Pareto dominates the allocation (x_i)? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

   Solution: (y_i) Pareto dominates (x_i) if $u_i(y_i) \geq u_i(x_i)$ for all i and $u_i(y_i) > u_i(x_i)$ for some i.

   (d) (3 points) What does it mean that the feasible allocation (x_i) is Pareto efficient? You can explain in words.

   Solution: (x_i) is Pareto efficient if no other feasible allocation (y_i) Pareto dominates it.

   (e) (7 points) Let $\{p, (x_i)\}$ be an Arrow-Debreu equilibrium. Prove that (x_i) is Pareto efficient.

   Solution: See the lecture note for the proof of the First Welfare Theorem.

2. Consider an economy with two agents indexed by $i = 1, 2$ and two goods indexed by $l = 1, 2$. The utility functions are

$$\begin{aligned} u_1(x_1, x_2) &= -\frac{1}{x_1} - \frac{1}{x_2}, \\ u_2(x_1, x_2) &= \frac{2}{3}\log x_1 + \frac{1}{3}\log x_2, \end{aligned}$$

   and the initial endowments are $e_1 = e_2 = (1, 6)$.

(a) (8 points) Is the initial endowment Pareto efficient? Answer yes or no, then explain why.

Solution: No (4 points). The marginal rate of substitution is

$$\mathrm{MRS}_1 = \frac{x_2^2}{x_1^2} = 6^2 = 36$$

for agent 1 and

$$\mathrm{MRS}_2 = 2\frac{x_2}{x_1} = 2 \times 6 = 12$$

for agent 2 at the initial allocation, which are not equal (2 points each for computing MRS).

(b) (6 points) Compute the Pareto efficient allocation in which agent 1 consumes 1 unit of good 1.

Solution: If $x_{11} = 1$, then $x_{21} = 1 + 1 - 1 = 1$. Let $x_{12} = t$. Then $x_{22} = 6 + 6 - t = 12 - t$. If the allocation is Pareto efficient, then the marginal rate of substitution must be the same across agents. Therefore

$$\frac{t^2}{1^2} = 2\frac{12-t}{1} \iff t^2 + 2t - 24 = 0 \iff (t-4)(t+6) = 0 \iff t = 4.$$

Therefore the allocation is $x_1 = (1, 4)$ and $x_2 = (1, 8)$.

(c) (6 points) Compute the competitive equilibrium with transfer payments when the allocation is the one in the previous question. (Normalize the price of good 1 to be 1, so $p_1 = 1$.)

Solution: In equilibrium, the marginal rate of substitution must be equal to the price ratio. Therefore

$$\frac{p_1}{p_2} = \frac{x_{12}^2}{x_{11}^2} \iff p_2 = (x_{11}/x_{12})^2 = \frac{1}{16}.$$

Transfers are $t_1 = 1/8$ and $t_2 = -1/8$.

3. Consider an economy with two goods indexed by $l = 1, 2$. Suppose that there is a small country (so it doesn't affect world prices) with two agents indexed by $i = 1, 2$ and endowments $e_1 = (3/2, 1)$, $e_2 = (1, 3/2)$. All agents have utility function

$$u(x_1, x_2) = x_1 x_2.$$

Below, always normalize the price of good 1 to be $p_1 = 1$.

(a) (5 points) Compute the autarky equilibrium allocation and price.

Solution: By symmetry, it must be $p_1 = p_2 = 1$ and $x_1 = x_2 = (5/4, 5/4)$.

(b) (7 points) Suppose that the country opens up to trade, and the price of good 2 changes to $p_2 = 1/2$. Compute the free trade allocation and utility and determine who gains/loses from trade.

Solution: Let $q = 1/2$ be the domestic price of good 2. Using the Cobb-Douglas formula, the demand of agent 1 is

$$(x_{11}, x_{12}) = \left(\frac{1}{2}\frac{3/2+q}{1}, \frac{1}{2}\frac{3/2+q}{q}\right) = (1, 2),$$

and the utility is

$$u_1 = 1 \times 2 = 2 > \left(\frac{5}{4}\right)^2 = \frac{25}{16}.$$

Similarly, the demand of agent 2 is

$$(x_{21}, x_{22}) = \left(\frac{1}{2}\frac{1+(3/2)q}{1}, \frac{1}{2}\frac{1+(3/2)q}{q}\right) = \left(\frac{7}{8}, \frac{7}{4}\right),$$

and the utility is

$$u_2 = \frac{7}{8} \times \frac{7}{4} = \frac{49}{32} < \frac{50}{32} = \frac{25}{16}.$$

Therefore agent 1 gains from trade and agent 2 loses from trade.

(c) (7 points) Suppose that the government imposes a tariff of $\tau = 1/4$ on the import of good 2, and the domestic price of good 2 becomes $q = p_2 + \tau = 1/2 + 1/4 = 3/4$. Let T be the tax revenue from the tariff, and suppose that the government gives out the tariff revenue equally to agents (so each agent gets $T/2$). Derive an equation that T satisfies.

Solution: Since agents have identical homothetic utility, the total demand depends only on aggregate wealth. Since the aggregate endowment is $(5/2, 5/2)$ and the tariff revenue is T, the aggregate demand for good 2 is

$$\frac{1}{2}\frac{(5/2)+(5/2)q+T}{q}.$$

The import of good 2 is thus

$$\frac{(5/2)(1+q)+T}{2q} - \frac{5}{2}.$$

Since import is taxed at rate $\tau = 1/4$, we must have

$$T = \tau\left(\frac{(5/2)(1+q)+T}{2q} - \frac{5}{2}\right).$$

(d) (6 points) Solve for the new allocation and show that all agents gain from trade.

Solution: Setting $\tau = 1/4$ and $q = 3/4$, the solution to the above equation is $T = 1/8$. After some algebra, the new allocation is

$$(x_{11}, x_{12}) = \left(\frac{37}{32}, \frac{37}{24}\right),$$
$$(x_{21}, x_{22}) = \left(\frac{35}{32}, \frac{35}{24}\right).$$

Clearly agent 1 is better off than agent 2 since consumption have common denominators but the numerators are larger. Furthermore,

$$\frac{35}{32} \times \frac{35}{24} > \frac{5}{4} \times \frac{5}{4} \iff \frac{7^2}{8 \times 6} = \frac{49}{48} > 1,$$

so agent 2 gains from trade. Therefore all agents gain from trade.

(e) (5 points) Propose a better policy than the government's. Compute the allocation and utility under your suggested policy.

Solution: Adopt free trade but introduce direct tax/subsidies to make the previous allocation just affordable, that is, determine t_i (tax on agent i) such that $p \cdot x_i^t = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i^t)$, where x_i^t is the consumption of agent i under the tariff regime. Since $p = (1, 1/2)$, $e_1 = (3/2, 1)$, and $x_1^t = (37/32, 37/24)$, it follows that $t_1 = 7/96$ and $t_2 = -7/96$. The agent 1's demand under this transfer is

$$\begin{aligned}(x_{11}, x_{12}) &= \left(\frac{1}{2}\frac{37/32 + (1/2)(37/24)}{1}, \frac{1}{2}\frac{37/32 + (1/2)(37/24)}{1/2}\right) \\ &= \frac{37}{32}\left(\frac{5}{6}, \frac{5}{3}\right).\end{aligned}$$

For agent 2, just change 37 to 35. To see that agent 1 gains from the free trade policy, note that

$$\frac{5}{6} \times \frac{5}{3} > 1 \times \frac{4}{3} \iff 25 > 24,$$

so free trade Pareto dominates the tariff regime.

4. Consider an economy with two countries, $i = A, B$, and two physical goods, $l = 1, 2$. The endowment is $e_A = (9, 2)$ and $e_B = (2, 9)$. The utility functions are

$$u_A(x_1, x_2) = \frac{2}{3}\log x_1 + \frac{1}{3}\log x_2,$$
$$u_B(x_1, x_2) = \frac{1}{3}\log x_1 + \frac{2}{3}\log x_2.$$

Suppose that there are transportation costs, and one third (1/3) of the exported goods perish by the time they reach the destination.

(a) (5 points) How many kinds of goods are there in the world? Answer the number and explain the reason.

Solution: 4, because there are two physical goods in two locations. (2 points for the answer, 3 point for the reason.)

(b) (5 points) Explain why a model of international trade with transportation costs can be regarded as a standard Arrow-Debreu model.

Solution: With transportation costs, goods must be distinguished by location as well as physical properties. Since transportation is a kind of technology that transforms a good in one country to some other goods in another country (with potentially many inputs such as fuel used for shipping, the labor service of a captain who operates the airplane or boat, etc.), a model of international trade with transportation costs can be interpreted as a standard general equilibrium model with production.

(c) (5 points) Assuming that country A imports good 2, what is its price? (Set the price of good 1 equal to 1.)

Solution: Relabel the goods by $l = 1, 2, 3, 4$, where good 1 is physical good 1 in country A, good 2 is physical good 2 in country A, and so on. Let (p_1, p_2, p_3, p_4) be the price. By symmetry, the equilibrium price must satisfy $p_1 = p_4 = 1$ and $p_2 = p_3$. If country B exports e units of good 4, it becomes $2e/3$ units of good 2. By doing so, the profit is $(2p_2/3 - p_4)e$. By profit maximization, we get

$$(2p_2/3 - p_4)e = 0 \iff p_2 = \frac{3}{2}p_4 = \frac{3}{2}.$$

(d) (10 points) Compute the free trade equilibrium. Make sure to compute all prices, consumption, and import/exports in each country.

Solution: By the Cobb-Douglas formula, the demand of country A, B is

$$\left(\frac{2}{3}\frac{10p_1 + 2p_2}{p_1}, \frac{1}{3}\frac{10p_1 + 2p_2}{p_2}, 0, 0\right) = (8, 8/3, 0, 0),$$
$$\left(0, 0, \frac{1}{3}\frac{2p_3 + 9p_4}{p_3}, \frac{2}{3}\frac{2p_3 + 9p_4}{2p_4}\right) = (0, 0, 8/3, 8).$$

Therefore the net import is $y_A = (-1, 2/3)$ for country A and $y_B = (2/3, -1)$ for country B.

You can detach this sheet and use as a scratch paper.