Mathematical Economics
Final Exam

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Name: _____

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base e $= 2.718281828\ldots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 180 minutes.
- This exam has 8 questions on 10 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	10	15	10	15	10	20	100
Score:									

1. (a) (2 points) What was the most interesting topic in this course? (Any nonempty answer gets full credit.)

   (b) (3 points) What is a put option? Explain.

   (c) (3 points) What is the difference between American and European options?

   (d) (2 points) What is the definition of a convex set?

2. Consider an economy with two goods and an agent with utility function

$$U(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2,$$

   where $0 < \alpha < 1$ is a parameter. Suppose that the agent has initial wealth w. Let the prices be $p_1 = 1$ and $p_2 = p$.

   (a) (3 points) Write down the Lagrangian for the utility maximization problem.

   (b) (3 points) Using the first-order condition, express x_1, x_2 using p and λ.

(c) (4 points) Express the demand using only p and w.

3. (10 points) Consider an economy with two goods and 100 agents. Agents have identical utility function

$$U(x_1, x_2) = \log x_1 + \log x_2.$$

Suppose that agent i's endowment is $(e_{i1}, e_{i2}) = (i, i(i+1))$, where $i = 1, 2, \ldots, 100$. Compute the equilibrium price and allocation (set $p_1 = 1$).

4. Consider an economy with two countries, $i = A, B$, and two physical goods, $l = 1, 2$. The endowment is $e_A = (10, 1)$ and $e_B = (1, 10)$. The utility function is

$$u(x_1, x_2) = \frac{1}{2} \log x_1 + \frac{1}{2} \log x_2$$

for all agents. Suppose that there are transportation costs, and 50% of the exported goods perish by the time they reach the destination.

(a) (2 points) How many kinds of goods are there in the world? Answer the number and explain the reason.

(b) (3 points) Explain why a model of international trade with transportation costs can be regarded as a standard Arrow-Debreu model.

(c) (5 points) Assuming that country A imports good 2, what is its price? (Set the price of good 1 equal to 1.)

(d) (5 points) Compute the free trade equilibrium. Make sure to compute all prices, consumption, and import/exports in each country.

5. (10 points) Consider a general equilibrium model with many countries and many agents. Suppose that initially all countries are in autarky. Explain how you can prove that by moving to free trade but with an appropriate tax/transfer within each country, the free trade allocation weakly Pareto dominates the autarky allocation. Make sure to list assumptions that you need and indicate how they are used.

6. Suppose that there are two assets, a stock and a (risk-free) bond. The current stock price is 100 and it can either go up to 120 or go down to 60 tomorrow. The risk-free interest rate is 5%. In answering the questions below, always use fractions.

   (a) (4 points) Let u, d stand for the up and down states and p_u, p_d be the state prices. Derive two equations that p_u, p_d satisfy.

   (b) (3 points) Compute p_u, p_d.

   (c) (2 points) Compute the price of a call option with strike 100.

(d) (2 points) Compute the price of a put option with strike 100.

(e) (2 points) Compute the price of a convertible bond that promises to pay 100 tomorrow. (A convertible bond is a promise to pay 100, with an option to deliver the stock instead.)

(f) (2 points) Compute the interest rate (yield) on the above convertible bond.

7. (10 points) What is the Mutual Fund Theorem? Explain the statement, assumptions, and practical implications.

8. Consider an economy with two periods, denoted by $t = 0, 1$, and three agents, denoted by $i = 1, 2, 3$. There are two states at $t = 1$, denoted by $s = 1, 2$. The two states occur with equal probability $\pi_1 = \pi_2 = 1/2$. Suppose that agent i's utility function is

$$U_i(x_0, x_1, x_2) = u_i(x_0) + \pi_1 u_i(x_1) + \pi_2 u_i(x_2),$$

where x_0, x_1, x_2 denote the consumption at $t = 0$ and states $s = 1, 2$, and the Bernoulli utility functions $u_i(x)$ are given by

$$u_1(x) = -\frac{4}{x},$$
$$u_2(x) = -\frac{4}{x-2},$$
$$u_3(x) = -\frac{4}{x+2}.$$

The initial endowments $e_i = (e_{i0}, e_{i1}, e_{i2})$ are given by

$$e_1 = (3, 5, 1),$$
$$e_2 = (3, 3, 3),$$
$$e_3 = (2, 4, 2).$$

(a) (2 points) What is the name of this type of utility functions?

(b) (2 points) For a given level of consumption, which agent is the most risk averse? Answer based on reasoning.

(c) (6 points) Normalize the price of $t = 0$ good to be $p_0 = 1$. Compute the equilibrium state prices p_1, p_2.

(d) (3 points) Compute the (gross) risk-free interest rate.

(e) (3 points) Consider an asset (stock) that pays out the aggregate endowment as dividend. Compute the ex-dividend stock price (the stock price excluding the dividend) at $t = 0$.

(f) (2 points) Compute the expected stock return at $t = 0$ and show that it is higher than the risk-free rate.

(g) (2 points) Compute the price at $t = 0$ of a put option written on a stock with strike price 9.

You can detach this sheet and use as a scratch paper.