Mathematical Economics
Final Exam

Prof. Alexis Akira Toda

December 14, 2017

Name: ____

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 180 minutes.
- This exam has 8 questions on 9 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	10	15	10	15	10	20	100
Score:									

1. (a) (2 points) What was the most interesting topic in this course? (Any nonempty answer gets full credit.)

   Solution: Write whatever you want.

   (b) (3 points) What is a put option? Explain.

   Solution: A put option is the right to sell a stock at a specified price (strike price).

   (c) (3 points) What is the difference between American and European options?

   Solution: European options can be exercised only at maturity. American options can be exercised at any time before maturity.

   (d) (2 points) What is the definition of a convex set?

   Solution: C is convex if for all $x, y \in C$ and $0 \le \alpha \le 1$, we have $(1-\alpha)x + \alpha y \in C$.

2. Consider an economy with two goods and an agent with utility function

$$U(x_1, x_2) = \alpha \log x_1 + (1-\alpha) \log x_2,$$

where $0 < \alpha < 1$ is a parameter. Suppose that the agent has initial wealth w. Let the prices be $p_1 = 1$ and $p_2 = p$.

   (a) (3 points) Write down the Lagrangian for the utility maximization problem.

   Solution:

$$L(x_1, x_2, \lambda) = \alpha \log x_1 + (1-\alpha) \log x_2 + \lambda(w - x_1 - px_2).$$

   (b) (3 points) Using the first-order condition, express x_1, x_2 using p and λ.

   Solution: By FOC, we have $\alpha/x_1 - \lambda = 0 \iff x_1 = \alpha/\lambda$ and $(1-\alpha)/x_2 = \lambda p \iff x_2 = (1-\alpha)/(\lambda p)$.

   (c) (4 points) Express the demand using only p and w.

   Solution: By complementary slackness, we have

$$w = x_1 + px_2 = \frac{\alpha}{\lambda} + \frac{1-\alpha}{\lambda} = \frac{1}{\lambda}.$$

   Therefore

$$\begin{aligned} x_1 &= \alpha w, \\ x_2 &= \frac{(1-\alpha)w}{p}. \end{aligned}$$

3. (10 points) Consider an economy with two goods and 100 agents. Agents have identical utility function
$$U(x_1, x_2) = \log x_1 + \log x_2.$$
Suppose that agent i's endowment is $(e_{i1}, e_{i2}) = (i, i(i+1))$, where $i = 1, 2, \ldots, 100$. Compute the equilibrium price and allocation (set $p_1 = 1$).

> **Solution:** Let $I = 100$. The aggregate endowment of good 1 is
> $$e_1 = \sum_{i=1}^{I} i = \frac{1}{2} I(I+1).$$
> The aggregate endowment of good 2 is
> $$e_2 = \sum_{i=1}^{I} i(i+1) = \frac{1}{3} I(I+1)(I+2).$$
> Since agents have identical Cobb-Douglas (hence homothetic) utility, we can treat the economy as a representative agent economy. Letting $p_2 = p$, since in equilibrium the price ratio equals the marginal rate of substitution, it follows that
> $$p = \frac{p_2}{p_1} = \frac{\partial U / \partial x_2}{\partial U / \partial x_1} = \frac{e_1}{e_2} = \frac{3}{2(I+2)} = \frac{3}{2 \cdot 102} = \frac{1}{68}.$$
> Agent i's wealth is then
> $$w_i = e_{i1} + p e_{i2} = \frac{1}{68} i(i+69).$$
> Using the Cobb-Douglas formula, agent i's demand is
> $$\begin{aligned} x_{i1} &= \frac{1}{2} w_i = \frac{1}{136} i(i+69), \\ x_{i2} &= \frac{1}{2} \frac{w_i}{p} = \frac{1}{2} i(i+69). \end{aligned}$$

4. Consider an economy with two countries, $i = A, B$, and two physical goods, $l = 1, 2$. The endowment is $e_A = (10, 1)$ and $e_B = (1, 10)$. The utility function is
$$u(x_1, x_2) = \frac{1}{2} \log x_1 + \frac{1}{2} \log x_2$$
for all agents. Suppose that there are transportation costs, and 50% of the exported goods perish by the time they reach the destination.

   (a) (2 points) How many kinds of goods are there in the world? Answer the number and explain the reason.

Solution: 4, because there are two physical goods in two locations. (2 points for the answer, 3 point for the reason.)

(b) (3 points) Explain why a model of international trade with transportation costs can be regarded as a standard Arrow-Debreu model.

Solution: With transportation costs, goods must be distinguished by location as well as physical properties. Since transportation is a kind of technology that transforms a good in one country to some other goods in another country (with potentially many inputs such as fuel used for shipping, the labor service of a captain who operates the airplane or boat, etc.), a model of international trade with transportation costs can be interpreted as a standard general equilibrium model with production.

(c) (5 points) Assuming that country A imports good 2, what is its price? (Set the price of good 1 equal to 1.)

Solution: Relabel the goods by $l = 1, 2, 3, 4$, where good 1 is physical good 1 in country A, good 2 is physical good 2 in country A, and so on. Let (p_1, p_2, p_3, p_4) be the price. By symmetry, the equilibrium price must satisfy $p_1 = p_4 = 1$ and $p_2 = p_3$. If country B exports e units of good 4, it becomes $0.5e$ units of good 2. By doing so, the profit is $(0.5p_2 - p_4)e$. By profit maximization, we get

$$(0.5p_2 - p_4)e = 0 \iff p_2 = 2p_4 = 2.$$

(d) (5 points) Compute the free trade equilibrium. Make sure to compute all prices, consumption, and import/exports in each country.

Solution: By the Cobb-Douglas formula, the demand of country A, B is

$$\left(\frac{10p_1 + p_2}{2p_1}, \frac{10p_1 + p_2}{2p_2}, 0, 0\right) = (6, 3, 0, 0),$$
$$\left(0, 0, \frac{p_3 + 10p_4}{2p_3}, \frac{p_3 + 10p_4}{2p_4}\right) = (0, 0, 3, 6).$$

Therefore the net import is $y_A = (-4, 2)$ for country A and $y_B = (2, -4)$ for country B.

5. (10 points) Consider a general equilibrium model with many countries and many agents. Suppose that initially all countries are in autarky. Explain how you can prove that by moving to free trade but with an appropriate tax/transfer within each country, the free trade allocation weakly Pareto dominates the autarky allocation. Make sure to list assumptions that you need and indicate how they are used.

Solution: See lecture note.

6. Suppose that there are two assets, a stock and a (risk-free) bond. The current stock price is 100 and it can either go up to 120 or go down to 60 tomorrow. The risk-free interest rate is 5%. In answering the questions below, always use fractions.

   (a) (4 points) Let u, d stand for the up and down states and p_u, p_d be the state prices. Derive two equations that p_u, p_d satisfy.

   Solution: Since the stock pays 120 in the up state and 90 in the down state, its price must be $120p_u + 60p_d$. Therefore
   $$120p_u + 60p_d = 100.$$
   Similarly, accounting the bond price, we obtain $105p_u + 105p_d = 100$.

   (b) (3 points) Compute p_u, p_d.

   Solution: Solving the above equations, we get $p_u = \frac{5}{7}$ and $p_d = \frac{5}{21}$.

   (c) (2 points) Compute the price of a call option with strike 100.

   Solution: The call option pays out $\max\{120 - 100, 0\} = 20$ in the up state and $\max\{60 - 100, 0\} = 0$ in the down state. Therefore its price is
   $$20p_u + 0p_d = \frac{100}{7}.$$

   (d) (2 points) Compute the price of a put option with strike 100.

   Solution: The put option pays out $\max\{100 - 120, 0\} = 0$ in the up state and $\max\{100 - 60, 0\} = 40$ in the down state. Therefore its price is
   $$0p_u + 40p_d = \frac{200}{21}.$$

   (e) (2 points) Compute the price of a convertible bond that promises to pay 100 tomorrow. (A convertible bond is a promise to pay 100, with an option to deliver the stock instead.)

   Solution: Since $60 < 100 < 120$, the convertible bond is exercised only in the down state, so it pays 100 in the up state and 60 in the down state. Therefore its price is
   $$100p_u + 60p_d = \frac{600}{7}.$$

(f) (2 points) Compute the interest rate (yield) on the above convertible bond.

Solution: Since the convertible bond promises to pay 100 and its price is 600/7, the gross interest rate is

$$100/(600/7) = \frac{7}{6} = 1 + \frac{1}{6}.$$

Therefore the yield is $100/6 = 16.6\%$.

7. (10 points) What is the Mutual Fund Theorem? Explain the statement, assumptions, and practical implications.

Solution: In a two period general equilibrium model with uncertainty, assuming that agents have hyperbolic relative risk aversion (HARA) preferences with identical parameter a, in equilibrium agents' consumption is spanned by only two vectors—the aggregate endowment and the vector of ones. Thus the agents' consumption can be replicated by only two assets, the aggregate stock market and the risk-free asset. This is called the mutual fund theorem. A practical implication of the theorem is that we should do passive investment—invest only in the market portfolio and risk-free asset and not active investment (*e.g.*, picking individual stocks).

8. Consider an economy with two periods, denoted by $t = 0, 1$, and three agents, denoted by $i = 1, 2, 3$. There are two states at $t = 1$, denoted by $s = 1, 2$. The two states occur with equal probability $\pi_1 = \pi_2 = 1/2$. Suppose that agent i's utility function is

$$U_i(x_0, x_1, x_2) = u_i(x_0) + \pi_1 u_i(x_1) + \pi_2 u_i(x_2),$$

where x_0, x_1, x_2 denote the consumption at $t = 0$ and states $s = 1, 2$, and the Bernoulli utility functions $u_i(x)$ are given by

$$\begin{aligned} u_1(x) &= -\frac{4}{x}, \\ u_2(x) &= -\frac{4}{x-2}, \\ u_3(x) &= -\frac{4}{x+2}. \end{aligned}$$

The initial endowments $e_i = (e_{i0}, e_{i1}, e_{i2})$ are given by

$$\begin{aligned} e_1 &= (3, 5, 1), \\ e_2 &= (3, 3, 3), \\ e_3 &= (2, 4, 2). \end{aligned}$$

(a) (2 points) What is the name of this type of utility functions?

Solution: Hyperbolic absolute risk aversion (HARA) utilities. Note that a HARA utility has a representation

$$u(x) = \frac{1}{a-1}(ax+b)^{1-\frac{1}{a}}.$$

The given functions correspond to $a = 1/2$ and $b = 0, -1, 1$.

(b) (2 points) For a given level of consumption, which agent is the most risk averse? Answer based on reasoning.

Solution: The absolute risk aversion (ARA) coefficient is given by

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax+b}.$$

Since $a > 0$ is common across agents, ARA is larger when b is smaller. Therefore agent 2 is the most risk averse.

(c) (6 points) Normalize the price of $t = 0$ good to be $p_0 = 1$. Compute the equilibrium state prices p_1, p_2.

Solution: Since agents have HARA utility with common a, by the argument in the lecture note, we can compute the equilibrium price through aggregation. Let e_s be the aggregate endowment in state s. Then

$$\begin{aligned} e_0 &= 3 + 3 + 2 = 8, \\ e_1 &= 5 + 3 + 4 = 12, \\ e_2 &= 1 + 3 + 2 = 6. \end{aligned}$$

Let $\pi_0 = 1$. Then the equilibrium price is given by

$$p_s = C\pi_s(ae_s + b)^{-\frac{1}{a}}$$

for some constant $C > 0$, where $b = \sum_{i=1}^{3} b_i = 0 + (-2) + 2 = 0$. Setting $a = 1/2$ and $b = 0$, it follows that

$$1 = p_0 = C(8/2)^{-2} = \iff C = 16,$$

so

$$\begin{aligned} p_1 &= 16\frac{1}{2}(12/2)^{-2} = \frac{2}{9}, \\ p_2 &= 16\frac{1}{2}(6/2)^{-2} = \frac{8}{9}. \end{aligned}$$

(d) (3 points) Compute the (gross) risk-free interest rate.

Solution: The risk-free asset pays 1 in states $s = 1, 2$. Therefore its price is

$$q = p_1 + p_2 = \frac{2}{9} + \frac{8}{9} = \frac{10}{9}.$$

The gross risk-free rate is

$$R_f = \frac{1}{q} = \frac{9}{10}.$$

(e) (3 points) Consider an asset (stock) that pays out the aggregate endowment as dividend. Compute the ex-dividend stock price (the stock price excluding the dividend) at $t = 0$.

Solution: The stock pays e_s in state s. Therefore its price is

$$q = p_1 e_1 + p_2 e_2 = \frac{2}{9} 12 + \frac{8}{9} 6 = 8.$$

(f) (2 points) Compute the expected stock return at $t = 0$ and show that it is higher than the risk-free rate.

Solution: Since each state occurs with probability 1/2, the expected stock return is

$$\mathrm{E}[R] = \frac{\mathrm{E}[e_s]}{q} = \frac{1}{8}\frac{1}{2}(12 + 6) = \frac{9}{8} > \frac{9}{10} = R_f.$$

(g) (2 points) Compute the price at $t = 0$ of a put option written on a stock with strike price 9.

Solution: Since the stock price (including the dividend) is e_s in state s, and $e_1 = 12 > 9$ and $e_2 = 6 < 9$, the put is exercised only in state 2. Letting $K = 9$ be the strike, the put price is therefore

$$q = p_2(K - e_2) = \frac{8}{9}(9 - 6) = \frac{8}{3}.$$

You can detach this sheet and use as a scratch paper.