Mathematical Economics
Midterm Exam

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November 16, 2017

Name: ____

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \ge 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \ge 0$.
- The exam time is 80 minutes.
- This exam has 4 questions on 7 pages excluding the cover page, for a total of 130 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	Total
Points:	25	35	35	35	130
Score:					

1. Consider an economy with I agents with utility functions (u_i). An *allocation* is denoted by (x_i), where x_i is the consumption bundle of agent i.

   (a) (5 points) What does it mean that an allocation (y_i) Pareto dominates the allocation (x_i)? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

   > **Solution:** (y_i) Pareto dominates (x_i) if
   >
   > 1. $u_i(y_i) \geq u_i(x_i)$ for all i, and
   > 2. $u_i(y_i) > u_i(x_i)$ for some i.
   >
   > That is, an allocation Pareto dominates another if the everybody weakly prefers the former and somebody strictly prefers the former.

   (b) (5 points) Let the initial endowment be (e_i). What does it mean that the allocation (x_i) is Pareto efficient? You can explain in words (maximum 4 points) or write down the precise mathematical definition.

   > **Solution:** (x_i) is Pareto efficient if
   >
   > 1. (x_i) is feasible, so $\sum_{i=1}^I x_i \leq \sum_{i=1}^I e_i$, and
   > 2. there are no other feasible allocation (y_i) that Pareto dominate (x_i).
   >
   > That is, an allocation is Pareto efficient if it is impossible to make somebody better off without making somebody worse off.

   (c) (5 points) What does the first welfare theorem say? You can explain in words.

   > **Solution:** The first welfare theorem says that the competitive equilibrium allocation is Pareto efficient, provided that utility functions are locally non-satiated. (1 point for referring to the equilibrium allocation, 3 points for saying Pareto efficient, and 1 point for mentioning local non-satiation.)

   (d) (10 points) How are the assumptions of the first and second welfare theorems different? You can explain in words.

   > **Solution:** For the first welfare theorem, the only assumption is local non-satiation (2 points). For the second welfare theorem, we assume that utility functions are continuous, quasi-concave, locally non-satiated, and that the allocation (x_i) is interior—that is, $x_i \gg 0$ for all i (2 points for each of these four assumptions).

2. Consider an economy with two countries ($i = 1, 2$) and two goods ($l = 1, 2$). Each country consists of a single agent type whose utility function is

$$u(x_1, x_2) = x_1 x_2.$$

Suppose that the labor endowments are $e_A = 10$, $e_B = 1$, and the vector of labor productivities are

$$(a_{A1}, a_{A2}) = (12, 6)$$
$$(a_{B1}, a_{B2}) = (4, 4).$$

(a) (10 points) Compute the autarky equilibrium in country A and the utility level. Note that you need to compute prices, wage, and allocations of goods and labor. Normalize the price of good 1 so that $p_1 = 1$.

Solution: By the zero profit condition we have $p_l a_{il} = w_i$, where w_i is the wage in country i. Since $p_1 = 1$, we obtain $w_A = a_{A1} = 12$ and $p_2 = w_A/a_{A2} = 2$. Since labor endowment is 10, using the Cobb-Douglas formula the demand of goods is

$$x_{A1} = \frac{1}{2}\frac{w_A e_A}{p_1} = 60,$$
$$x_{A2} = \frac{1}{2}\frac{w_A e_A}{p_2} = 30.$$

Letting e_{il} be the labor employment in country i, industry l. Solving $x_{il} = a_{il}e_{il}$ for $i = A$ and $l = 1, 2$, we obtain $e_{A1} = 5$, $e_{A2} = 5$. The utility level is $U_A^a = x_{A1}x_{A2} = 1800$.

(b) (5 points) Repeat the previous question for country B.

Solution: $p_1 = p_2 = 1$, $w_B = 4$,

$$x_{B1} = \frac{1}{2}\frac{w_B e_B}{p_1} = 2,$$
$$x_{B2} = \frac{1}{2}\frac{w_B e_B}{p_2} = 2,$$

$e_{B1} = e_{B2} = 1/2$, utility level $U_B^a = 4$.

(c) (10 points) Compute the free trade equilibrium.

Solution: Since $12/4 > 6/4$, A has the comparative advantage in producing good 1. Therefore B will specialize in good 2. Since country A is large and more productive, in equilibrium we can guess that A will produce both goods. In this case prices are $p_1 = 1$, $p_2 = 2$, $w_A = 12$ and the demand and utility are the same as in autarky. For country B, the zero profit condition is $p_2 a_{B2} = w_B$, so $w_B = 8$. The demand in country B is

$$x_{B1} = \frac{1}{2}\frac{w_B e_B}{p_1} = 4,$$
$$x_{B2} = \frac{1}{2}\frac{w_B e_B}{p_2} = 2,$$

and utility is $U_B^f = 8$. Labor allocation in country B is $e_{B1} = 0$ and $e_{B2} = 1$. Since country A produces all of good 1 consumed in the world, we have $x_{A1} + x_{B1} = a_{A1}e_{A1}$. Therefore $e_{A1} = 64/12 = 16/3$, so $e_{B1} = 10 - 16/3 = 14/3$.

(d) (10 points) Explain why during international conflicts, large/developed countries often try to impose an embargo on small/developing countries. (Imagine the U.S.-Japan relationship before WWII or U.S.-North Korea now.)

Solution: In Ricardo's international trade model, if a country (say A) is large and/or more productive, it may be the case that it produces all goods both in autarky as well as under free trade. In this case since the relative prices do not change, the utility is the same. On the other hand, the small and/or less productive country (say B) gains from free trade. Since A has nothing to lose by shutting down trade but B does, it makes sense for A to impose an embargo on B during an international conflict to punish the other without hurting yourself.

3. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility functions are

$$\begin{aligned} u_1(x_1, x_2) &= x_1^2 x_2, \\ u_2(x_1, x_2) &= x_1 x_2^2, \\ u_3(x_1, x_2) &= x_1 x_2. \end{aligned}$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (18, 6)$. In answering questions below, in order to make the notation consistent use x_{il} for consumption of good l by agent i. (So x_{12} is consumption of good 2 by agent 1, for example.) Also, use $p_1 = 1$ and $p_2 = p$ for the prices.

(a) (5 points) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

Solution: Using the Cobb-Douglas formula, the demand of agents 1, 2 are

$$\begin{aligned} (x_{11}, x_{12}) &= \left(\frac{2(3+3p)}{3}, \frac{3+3p}{3p} \right) = (2 + 2p, 1 + 1/p), \\ (x_{21}, x_{22}) &= \left(\frac{3+3p}{3}, \frac{2(3+3p)}{3p} \right) = (1 + p, 2 + 2/p). \end{aligned}$$

Therefore by market clearing we have

$$(2 + 2p) + (1 + p) = 3 + 3 \iff p = 1.$$

The demand is $(x_{11}, x_{12}) = (4, 2)$, $(x_{21}, x_{22}) = (2, 4)$. The utility level is $u_1^a = 4^2 \cdot 2 = 32$ and $u_2^a = 2 \cdot 4^2 = 32$.

(b) (10 points) Compute the free trade equilibrium price and allocation.

Solution: Using the Cobb-Douglas formula, the demand of agent 3 is

$$(x_{31}, x_{32}) = \left(\frac{18+6p}{2}, \frac{18+6p}{2p}\right) = (9+3p, 3+9/p).$$

Hence by market clearing, we have

$$(2+2p) + (1+p) + (9+3p) = 3+3+18 \iff p = 2.$$

Again using the Cobb-Douglas formula, the consumption of each agents are $(x_{11}, x_{12}) = (6, 3/2)$, $(x_{21}, x_{22}) = (3, 3)$, and $(x_{31}, x_{32}) = (15, 15/2)$.

(c) (5 points) Compute the utility level of each agent and determine who gained from trade and who lost.

Solution: Utility levels are

$$\begin{aligned}
u_1^f &= 6^2 \cdot 3/2 = 54 > 32 = u_1^a, \\
u_2^f &= 3 \cdot 3^2 = 27 < 32 = u_2^a, \\
u_3^f &= 15 \cdot 15/2 = \frac{225}{2} > 108 = 18 \cdot 6 = u_3^a.
\end{aligned}$$

Therefore agents 1 and 3 gained from trade and agent 2 lost.

(d) (15 points) Find a tax scheme in country A such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

Solution: Consider a hypothetical economy in which agents start with the autarky allocation $e_1' = (4, 2)$, $e_2' = (2, 4)$, and $e_3' = (18, 6)$. Using the Cobb-Douglas formula, the demand is

$$\begin{aligned}
(x_{11}, x_{12}) &= \left(\frac{2(4+2p)}{3}, \frac{4+2p}{3p}\right), \\
(x_{21}, x_{22}) &= \left(\frac{2+4p}{3}, \frac{2(2+4p)}{3p}\right), \\
(x_{31}, x_{32}) &= \left(\frac{18+6p}{2}, \frac{18+6p}{2p}\right).
\end{aligned}$$

The market clearing condition is

$$\frac{2(4+2p)}{3} + \frac{2+4p}{3} + \frac{18+6p}{2} = 4+2+18 \iff p = \frac{35}{17}.$$

This price should be the free trade price after the tax and transfer. To make the autarky equilibrium allocation just affordable, taxes should be set such that $p \cdot x_i^a = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i^a)$. Therefore

$$t_1 = (1, 35/17) \cdot (3 - 4, 3 - 2) = -1 + \frac{35}{17} = \frac{18}{17},$$
$$t_2 = (1, 35/17) \cdot (3 - 2, 3 - 4) = 1 - \frac{35}{17} = -\frac{18}{17}.$$

With these tax/transfers and price, since the autarky allocation is affordable, the equilibrium allocation Pareto dominates the autarky allocation. (5 points for the candidate price, 5 points for the tax/transfers, and 5 points for the explanation why it is Pareto improving.)

4. Consider Ricardo's international trade model with two countries, $i = A, B$, and L consumption goods indexed by $l = 1, \dots, L$. Let $a_{il} > 0$ be the labor productivity of country i when producing good l. Assume that the values $\{a_{Al}/a_{Bl}\}_{l=1}^L$ are all distinct.

   (a) (5 points) Define the notion of comparative advantage.

   Solution: We say that country A has a comparative advantage over country B in producing good l rather than l' if $a_{Al}/a_{Bl} > a_{Al'}/a_{Bl'}$.

   (b) (15 points) Let $w_i > 0$ be the wage rate in country i and $p_l > 0$ be the price of good l. In equilibrium, prove that $p_l a_{il} \le w_i$, with equality if country i produces good l.

   Solution: Suppose on the contrary that $p_l a_{il} > w_i$. Then if a firm in country i employs labor $x > 0$ to produce good l, the profit is

   $$p_l a_{il} x - w_i x = (p_l a_{il} - w_i) x > 0.$$

   Letting $x \to \infty$, the profit diverges to ∞, which is incompatible with profit maximization. Therefore $p_l a_{il} \le w_i$. If country i produces good l, then the profit cannot be negative, so $p_l a_{il} - w_i \ge 0$. Therefore $p_l a_{il} = w_i$.

   (c) (15 points) Let $L = \{1, \dots, L\}$ be the set of goods. Let L_i ($i = A, B$) be the set of goods produced by country i in equilibrium. Prove that $L_A \cup L_B = L$ and $L_A \cap L_B$ is either empty or consists of a single element.

   Solution: Since in equilibrium all goods must be produced, we have $L_A \cup L_B = L$. Suppose that $L_A \cap L_B$ contains at least two elements l, l'. Then by the result of the previous question, we have $p_l a_{il} = w_i$ and $p_{l'} a_{il'} = w_i$ for $i = A, B$. Therefore

   $$\frac{a_{Al}}{a_{Bl}} = \frac{a_{Al'}}{a_{Bl'}},$$

   which is a contradiction.

You can detach this sheet and use as a scratch paper.