Econ 113 Mathematical Economics
Final Exam

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Name: _____

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, i.e., base e $= 2.718281828\ldots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 180 minutes.
- This exam has 6 questions on 11 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	6	Total
Points:	10	15	15	15	15	30	100
Score:							

1. The hyperbolic absolute risk aversion (HARA) utility function satisfies

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax+b},$$

where a, b are constants and we only consider the range of x such that $ax + b > 0$.

(a) (3 points) Derive the functional form of u (up to a monotonic transformation) when $a = 0$.

(b) (3 points) Derive the functional form of u (up to a monotonic transformation) when $a = 1$.

(c) (4 points) Derive the functional form of u (up to a monotonic transformation) when $a \neq 0, 1$.

2. Consider an economy with two goods and I agents. Agent i has endowment $(e_{i1}, e_{i2}) \gg 0$ and utility function

$$u_i(x_1, x_2) = \alpha_i \log x_1 + (1 - \alpha_i) \log x_2,$$

where $0 < \alpha_i < 1$. Let the prices be $p_1 = 1$ and $p_2 = p$.

(a) (3 points) Write down the Lagrangian for the utility maximization problem of agent i.

(b) (3 points) Using the first-order condition, express agent i's demand using p and the Lagrange multiplier.

(c) (4 points) Express agent i's demand using only p and other exogenous parameters.

(d) (5 points) Does this economy has an equilibrium? If so, is it unique?

3. (15 points) State and prove the First Welfare Theorem under appropriate assumptions.

4. Consider an economy with two agents indexed by $i = 1, 2$ and two goods indexed by $l = 1, 2$. The utility functions are

$$u_1(x_1, x_2) = -\frac{1}{x_1} - \frac{1}{x_2},$$
$$u_2(x_1, x_2) = \frac{2}{3}\log x_1 + \frac{1}{3}\log x_2,$$

and the initial endowments are $e_1 = e_2 = (1, 6)$.

(a) (5 points) Is the initial endowment Pareto efficient? Answer yes or no, then explain why.

(b) (5 points) Compute the Pareto efficient allocation in which agent 1 consumes 1 unit of good 1.

(c) (5 points) Compute the competitive equilibrium with transfer payments when the allocation is the one in the previous question. (Normalize the price of good 1 to be 1, so $p_1 = 1$.)

5. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility functions are

$$u_1(x_1, x_2) = x_1^2 x_2,$$
$$u_2(x_1, x_2) = x_1 x_2^2,$$
$$u_3(x_1, x_2) = x_1 x_2.$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (6, 18)$. In answering questions below, in order to make the notation consistent use x_{il} for consumption of good l by agent i. (So x_{12} is consumption of good 2 by agent 1, for example.) Also, use $p_1 = 1$ and $p_2 = p$ for the prices.

(a) (3 points) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

(b) (4 points) Compute the free trade equilibrium price and allocation.

(c) (3 points) Compute the utility level of each agent and determine who gained from trade and who lost.

(d) (5 points) Find a tax scheme in country A such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

6. Consider the binomial option pricing model discussed in the lectures. Time is denoted by $t = 0, 1, \ldots, T$. The gross risk-free rate is constant at $R > 0$. Each period, the stock can go up or down, so

$$S_{t+1} = \begin{cases} US_t & \text{if stock goes up,} \\ DS_t & \text{if stock goes down,} \end{cases}$$

where $U > R > D$. Suppose the initial stock price $S_0 > 0$ is given and consider an American put option with strike price K and expiration T. Recall that a put option is the right to sell a stock at the strike price, and "American" means that the option can be exercised any time until expiration.

(a) (4 points) Suppose for simplicity that $T = 1$. Let u, d stand for the up and down states and p_u, p_d be the state prices. Derive two equations that p_u, p_d satisfy.

(b) (3 points) Compute p_u, p_d.

(c) (3 points) Compute the price of the American put option.

(d) (10 points) What is the relation between the interest rate R and the put option price? Is put price increasing, decreasing, or ambiguous in R?

(e) (10 points) How does your answer to the previous question change for general expiration date T?

You can detach this sheet and use as a scratch paper.