Econ 113 Mathematical Economics
Final Exam

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Name: ______________________________

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, i.e., base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \ge 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \ge 0$.
- The exam time is 180 minutes.
- This exam has 6 questions on 8 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	6	Total
Points:	10	15	15	15	15	30	100
Score:							

1. The hyperbolic absolute risk aversion (HARA) utility function satisfies

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax+b},$$

where a, b are constants and we only consider the range of x such that $ax + b > 0$.

(a) (3 points) Derive the functional form of u (up to a monotonic transformation) when $a = 0$.

> **Solution:** $u(x) = -b\mathrm{e}^{-x/b}$, see lecture note for derivation.

(b) (3 points) Derive the functional form of u (up to a monotonic transformation) when $a = 1$.

> **Solution:** $u(x) = \log(x + b)$, see lecture note for derivation.

(c) (4 points) Derive the functional form of u (up to a monotonic transformation) when $a \neq 0, 1$.

> **Solution:** $u(x) = \frac{1}{a-1}(ax + b)^{1-1/a}$, see lecture note for derivation.

2. Consider an economy with two goods and I agents. Agent i has endowment $(e_{i1}, e_{i2}) \gg 0$ and utility function

$$u_i(x_1, x_2) = \alpha_i \log x_1 + (1 - \alpha_i) \log x_2,$$

where $0 < \alpha_i < 1$. Let the prices be $p_1 = 1$ and $p_2 = p$.

(a) (3 points) Write down the Lagrangian for the utility maximization problem of agent i.

> **Solution:**
>
> $$L(x_1, x_2, \lambda) = \alpha \log x_1 + (1 - \alpha) \log x_2 + \lambda(w - x_1 - px_2),$$
>
> where $\alpha = \alpha_i$ and $w = w_i = e_{i1} + pe_{i2}$.

(b) (3 points) Using the first-order condition, express agent i's demand using p and the Lagrange multiplier.

> **Solution:** By FOC, we have $\alpha/x_1 - \lambda = 0 \iff x_1 = \alpha/\lambda$ and $(1-\alpha)/x_2 = \lambda p \iff x_2 = (1 - \alpha)/(\lambda p)$.

(c) (4 points) Express agent i's demand using only p and other exogenous parameters.

Solution: By complementary slackness, we have

$$w = x_1 + px_2 = \frac{\alpha}{\lambda} + \frac{1-\alpha}{\lambda} = \frac{1}{\lambda}.$$

Therefore

$$\begin{aligned} x_1 &= \alpha w = \alpha_i(e_{i1} + pe_{i2}), \\ x_2 &= \frac{(1-\alpha)w}{p} = \frac{(1-\alpha_i)(e_{i1} + pe_{i2})}{p}. \end{aligned}$$

(d) (5 points) Does this economy has an equilibrium? If so, is it unique?

Solution: Using the above demand formula, market clearing condition for good 1 is

$$0 = \sum_{i=1}^{I}(x_{i1} - e_{i1}) = \sum_{i=1}^{I}(\alpha_i(e_{i1} + pe_{i2}) - e_{i1}) \iff p = \frac{\sum_{i=1}^{I}(1-\alpha_i)e_{i1}}{\sum_{i=1}^{I}\alpha_i e_{i2}},$$

which is positive and unique. The market for good 2 automatically clears by the Walras law. Therefore there exists an equilibrium and it is unique.

3. (15 points) State and prove the First Welfare Theorem under appropriate assumptions.

Solution: See lecture note.

4. Consider an economy with two agents indexed by $i = 1, 2$ and two goods indexed by $l = 1, 2$. The utility functions are

$$\begin{aligned} u_1(x_1, x_2) &= -\frac{1}{x_1} - \frac{1}{x_2}, \\ u_2(x_1, x_2) &= \frac{2}{3}\log x_1 + \frac{1}{3}\log x_2, \end{aligned}$$

and the initial endowments are $e_1 = e_2 = (1, 6)$.

(a) (5 points) Is the initial endowment Pareto efficient? Answer yes or no, then explain why.

Solution: No (4 points). The marginal rate of substitution is

$$\mathrm{MRS}_1 = \frac{x_2^2}{x_1^2} = 6^2 = 36$$

for agent 1 and

$$\mathrm{MRS}_2 = 2\frac{x_2}{x_1} = 2 \times 6 = 12$$

for agent 2 at the initial allocation, which are not equal (2 points each for computing MRS).

(b) (5 points) Compute the Pareto efficient allocation in which agent 1 consumes 1 unit of good 1.

Solution: If $x_{11} = 1$, then $x_{21} = 1 + 1 - 1 = 1$. Let $x_{12} = t$. Then $x_{22} = 6 + 6 - t = 12 - t$. If the allocation is Pareto efficient, then the marginal rate of substitution must be the same across agents. Therefore

$$\frac{t^2}{1^2} = 2\frac{12-t}{1} \iff t^2 + 2t - 24 = 0 \iff (t-4)(t+6) = 0 \iff t = 4.$$

Therefore the allocation is $x_1 = (1, 4)$ and $x_2 = (1, 8)$.

(c) (5 points) Compute the competitive equilibrium with transfer payments when the allocation is the one in the previous question. (Normalize the price of good 1 to be 1, so $p_1 = 1$.)

Solution: In equilibrium, the marginal rate of substitution must be equal to the price ratio. Therefore

$$\frac{p_1}{p_2} = \frac{x_{12}^2}{x_{11}^2} \iff p_2 = (x_{11}/x_{12})^2 = \frac{1}{16}.$$

Transfers are $t_1 = 1/8$ and $t_2 = -1/8$.

5. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility functions are

$$u_1(x_1, x_2) = x_1^2 x_2,$$
$$u_2(x_1, x_2) = x_1 x_2^2,$$
$$u_3(x_1, x_2) = x_1 x_2.$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (6, 18)$. In answering questions below, in order to make the notation consistent use x_{il} for consumption of good l by agent i. (So x_{12} is consumption of good 2 by agent 1, for example.) Also, use $p_1 = 1$ and $p_2 = p$ for the prices.

(a) (3 points) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

Solution: Using the Cobb-Douglas formula, the demand of agents 1, 2 are

$$(x_{11}, x_{12}) = \left(\frac{2(3+3p)}{3}, \frac{3+3p}{3p}\right) = (2 + 2p, 1 + 1/p),$$
$$(x_{21}, x_{22}) = \left(\frac{3+3p}{3}, \frac{2(3+3p)}{3p}\right) = (1 + p, 2 + 2/p).$$

Therefore by market clearing we have

$$(2+2p)+(1+p)=3+3 \iff p=1.$$

The demand is $(x_{11},x_{12})=(4,2)$, $(x_{21},x_{22})=(2,4)$. The utility level is $u_1^a=4^2\cdot 2=32$ and $u_2^a=2\cdot 4^2=32$.

(b) (4 points) Compute the free trade equilibrium price and allocation.

Solution: Using the Cobb-Douglas formula, the demand of agent 3 is

$$(x_{31},x_{32})=\left(\frac{6+18p}{2},\frac{6+18p}{2p}\right)=(3+9p,9+3/p).$$

Hence by market clearing, we have

$$(2+2p)+(1+p)+(3+9p)=3+3+6 \iff p=\frac{1}{2}.$$

Again using the Cobb-Douglas formula, the consumption of each agents are $(x_{11},x_{12})=(3,3)$, $(x_{21},x_{22})=(3/2,6)$, and $(x_{31},x_{32})=(15/2,15)$.

(c) (3 points) Compute the utility level of each agent and determine who gained from trade and who lost.

Solution: Utility levels are

$$\begin{aligned}
u_1^f &= 3^2\cdot 3=27<32=u_1^a,\\
u_2^f &= 3/2\cdot 6^2=54>32=u_2^a,\\
u_3^f &= 15/2\cdot 15=\frac{225}{2}>108=6\cdot 18=u_3^a.
\end{aligned}$$

Therefore agents 2 and 3 gained from trade and agent 1 lost.

(d) (5 points) Find a tax scheme in country A such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

Solution: Consider a hypothetical economy in which agents start with the autarky allocation $e_1'=(4,2)$, $e_2'=(2,4)$, and $e_3'=(6,18)$. Using the Cobb-Douglas formula, the demand is

$$\begin{aligned}
(x_{11},x_{12}) &= \left(\frac{2(4+2p)}{3},\frac{4+2p}{3p}\right),\\
(x_{21},x_{22}) &= \left(\frac{2+4p}{3},\frac{2(2+4p)}{3p}\right),\\
(x_{31},x_{32}) &= \left(\frac{6+18p}{2},\frac{6+18p}{2p}\right).
\end{aligned}$$

The market clearing condition is

$$\frac{2(4+2p)}{3}+\frac{2+4p}{3}+\frac{6+18p}{2}=4+2+6 \iff p=\frac{17}{22}.$$

This price should be the free trade price after the tax and transfer. To make the autarky equilibrium allocation just affordable, taxes should be set such that $p \cdot x_i^a = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i^a)$. Therefore

$$\begin{aligned} t_1 &= (1, 17/22) \cdot (3-4, 3-2) = -1 + \frac{17}{22} = -\frac{5}{22}, \\ t_2 &= (1, 17/22) \cdot (3-2, 3-4) = 1 - \frac{17}{22} = \frac{5}{22}. \end{aligned}$$

With these tax/transfers and price, since the autarky allocation is affordable, the equilibrium allocation Pareto dominates the autarky allocation.

6. Consider the binomial option pricing model discussed in the lectures. Time is denoted by $t = 0, 1, \dots, T$. The gross risk-free rate is constant at $R > 0$. Each period, the stock can go up or down, so

$$S_{t+1} = \begin{cases} US_t & \text{if stock goes up,} \\ DS_t & \text{if stock goes down,} \end{cases}$$

where $U > R > D$. Suppose the initial stock price $S_0 > 0$ is given and consider an American put option with strike price K and expiration T. Recall that a put option is the right to sell a stock at the strike price, and "American" means that the option can be exercised any time until expiration.

(a) (4 points) Suppose for simplicity that $T = 1$. Let u, d stand for the up and down states and p_u, p_d be the state prices. Derive two equations that p_u, p_d satisfy.

Solution: Since the stock pays US_0 in the up state and DS_0 in the down state, its price must be $US_0 p_u + DS_0 p_d$. Therefore

$$S_0 = US_0 p_u + DS_0 p_d \iff Up_u + Dp_d = 1.$$

Similarly, accounting for the bond price, we obtain $1 = Rp_u + Rp_d$.

(b) (3 points) Compute p_u, p_d.

Solution: Expressing in matrix form, we obtain

$$\begin{aligned} &\begin{bmatrix} U & D \\ R & R \end{bmatrix} \begin{bmatrix} p_u \\ p_d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \iff &\begin{bmatrix} p_u \\ p_d \end{bmatrix} = \frac{1}{R(U-D)} \begin{bmatrix} R & -D \\ -R & U \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{R(U-D)} \begin{bmatrix} R-D \\ U-R \end{bmatrix} \end{aligned}$$

(c) (3 points) Compute the price of the American put option.

Solution: The put payoff is $K - S$ if exercised. Rational agents exercise options only if $K - S > 0$. Therefore the price of the American put option is

$$P = \max\{K - S_0, p_u P(u) + p_d P(d)\},$$

where

$$\begin{aligned} P(u) &= \max\{0, K - US_0\}, \\ P(d) &= \max\{0, K - DS_0\} \end{aligned}$$

are the put payoffs in states u, d.

(d) (10 points) What is the relation between the interest rate R and the put option price? Is put price increasing, decreasing, or ambiguous in R?

Solution: Let $p = \frac{R-D}{U-D}$ be the risk-neutral probability of state u. Then we can write

$$\begin{aligned} P &= \max\{K - S_0, p_u P(u) + p_d P(d)\} \\ &= \max\left\{K - S_0, \frac{1}{R}(pP(u) + (1-p)P(d))\right\} \\ &= \max\left\{K - S_0, \frac{1}{R}(P(u) + (1-p)(P(d) - P(u)))\right\}. \end{aligned}$$

Clearly $P(d) \geq P(u)$ since $U > D$, so $P(d) - P(u) \geq 0$. Furthermore, $1/R$ and

$$1 - p = \frac{U - R}{U - D}$$

are decreasing in R. Therefore P is decreasing in R.

(e) (10 points) How does your answer to the previous question change for general expiration date T?

Solution: The answer remains the same. To see this, let P_t be the put option price at t and let $P_{t+1}(u), P_{t+1}(d)$ be the put option prices at $t+1$ when the stock goes up or down. Since a multi-period model is just a repetition of a two period model, we can prove $P_{t+1}(u) \leq P_{t+1}(d)$ by induction. Since

$$P_t = \max\left\{K - S_0, \frac{1}{R}(P_{t+1}(u) + (1-p)(P(d)_{t+1} - P_{t+1}(u)))\right\}$$

and the terminal payoffs do not depend on R, it follows from induction that P_t is decreasing in R.

You can detach this sheet and use as a scratch paper.