Econ 113 Mathematical Economics
Midterm Exam

Prof. Alexis Akira Toda

November 14, 2019

Name: ____

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, i.e., base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \ge 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \ge 0$.
- The exam time is 80 minutes.
- This exam has 4 questions on 7 pages excluding the cover page, for a total of 120 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	Total
Points:	30	30	30	30	120
Score:					

1. Consider an economy with two physical goods (apples and bananas) and two periods denoted by $t = 1, 2$. There is only one agent (type) with utility

$$U(x_1, y_1, x_2, y_2) = u(x_1, y_1) + \beta u(x_2, y_2),$$

where

$$u(x, y) = \alpha \log x + (1 - \alpha) \log y$$

is the period utility function, $\beta > 0$ is the discount factor, and x_t, y_t denote the consumption of apples and bananas in period t. Let the endowment of apples and bananas in period t be a_t and b_t.

(a) (10 points) Compute the gross real interest rate in units of apples.

Solution: In general, let p_t be the price of some physical good in period t. By selling one unit of good at $t = 1$, you can buy p_1/p_2 units of future contracts. Therefore the gross real interest rate is p_1/p_2.

Now let p_t^a be the price of an apple in period t. By the first-order condition (and using market clearing), we have

$$\lambda p_1^a = \frac{\partial U}{\partial x_1} = \frac{\alpha}{a_1},$$
$$\lambda p_2^a = \frac{\partial U}{\partial x_2} = \beta \frac{\alpha}{a_2},$$

Dividing the first equation by the second, we obtain the gross apple interest rate

$$\frac{p_1^a}{p_2^a} = \frac{1}{\beta} \frac{a_2}{a_1}.$$

(b) (10 points) Compute the gross real interest rate in units of bananas.

Solution: By the exact same argument, we obtain the gross banana interest rate

$$\frac{p_1^b}{p_2^b} = \frac{1}{\beta} \frac{b_2}{b_1}.$$

(c) (10 points) Under what condition do the two interest rates coincide? Is it surprising that the two rates are distinct except a very special case?

Solution: The two rates coincide if and only if

$$\frac{1}{\beta} \frac{a_2}{a_1} = \frac{1}{\beta} \frac{b_2}{b_1} \iff \frac{b_1}{a_1} = \frac{b_2}{a_2},$$

that is, when the endowment ratios between the two goods are the same in each period.

It is not surprising that the real interest rate depends on the choice of the good because it is a relative price. For examples, smart phones have advanced so much during the past decade, whereas rice production hasn't changed so much, so "smart phone interest rate" should be much higher than "rice interest rate".

2. (a) (10 points) What is the definition of local nonsatiation? Provide both intuitive and formal definitions.

Solution: Intuitively, a utility function is locally nonsatiated if no matter what the agent is consuming now, there exists an arbitrarily close consumption bundle that gives strictly higher utility.

Formally, a utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ is said to be locally nonsatiated if for any $x \in \mathbb{R}^L_+$ and $\epsilon > 0$, there exists $y \in \mathbb{R}^L_+$ such that $\|y - x\| < \epsilon$ and $u(y) > u(x)$.

(b) (10 points) What is the definition of Pareto efficiency?

Solution: An allocation (x_i) is called feasible if $\sum_{i=1}^I (x_i - e_i) \leq 0$, where e_i is the endowment of agent i. An allocation (x_i) is Pareto efficient if there exist no other feasible allocation (y_i) such that $u_i(y_i) \geq u_i(x_i)$ for all i and $u_i(y_i) > u_i(x_i)$ for some i.

(c) (10 points) First Welfare Theorem says that if utility functions are locally nonsatiated, then a competitive equilibrium allocation is Pareto efficient. Construct an example such that utility functions are not necessarily locally nonsatiated and the competitive equilibrium is Pareto inefficient.

Solution: Consider the following example. There are two goods and two agents. Endowments are $e_1 = e_2 = (1, 1)$. Utility functions are $u_1(x_1, x_2) = x_1 x_2$ and $u_2(x_1, x_2) = 0$. Consider the price vector $p = (1, 1)$. Then the initial endowment is a competitive equilibrium because agent 1's demand is $x_1 = (1, 1)$ and agent 2's demand is arbitrary (so we can take $x_1 = (1, 1)$). This allocation is inefficient because it is Pareto dominated by the feasible allocation $(y_1, y_2) = ((2, 2), (0, 0))$.

3. Consider an economy with two countries ($i = 1, 2$) and two goods ($l = 1, 2$). Each country consists of a single agent type whose utility function is

$$u(x_1, x_2) = x_1 x_2.$$

Suppose that the labor endowments are $e_A = 1$, $e_B = e$, and the vector of labor productivities are

$$(a_{A1}, a_{A2}) = (2, 1)$$
$$(a_{B1}, a_{B2}) = (1, 1).$$

Below, normalize the price of good 1 so that $p_1 = 1$.

(a) (5 points) Define the notion of comparative advantage.

Solution: We say that country A has a comparative advantage over country B in producing good l rather than l' if $a_{Al}/a_{Bl} > a_{Al'}/a_{Bl'}$.

(b) (10 points) Under what condition on e is there a free trade equilibrium in which country A produces both goods? Compute the equilibrium.

Solution: Using the Cobb-Douglas formula, demand is

$$\begin{aligned}(x_{A1}, x_{A2}) &= \left(\frac{w_A}{2}, \frac{w_A}{2p}\right),\\ (x_{B1}, x_{B2}) &= \left(\frac{w_B e}{2}, \frac{w_B e}{2p}\right).\end{aligned}$$

Country A produces both goods if it is large enough relative to B, so when e is small enough. Let $p_2 = p$ and w_i be the wage in country i. Zero profit conditions imply

$$\begin{aligned}p_1 a_{A1} - w_A = 0 &\iff w_A = a_{A1} = 2,\\ p_2 a_{A2} - w_A = 0 &\iff p = w_A/a_{A2} = 2.\end{aligned}$$

Since country A has comparative advantage in good 1, B produces only good 2. Hence zero profit condition implies

$$p_2 a_{B2} - w_B = 0 \iff w_B = p a_{B2} = 2.$$

Since country A produces the global supply of good 1, market clearing implies

$$x_1^A + x_1^B = a_{A1} e_{A1} \iff e_{A1} = \frac{1+e}{2}.$$

Since by assumption country A produces both goods, by labor market clearing it must be

$$\frac{1+e}{2} < 1 \iff e < 1.$$

In this case in equilibrium the price is $(p_1, p_2) = (1, 2)$, consumption is

$$(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = \left(1, \frac{1}{2}, e, \frac{e}{2}\right),$$

and labor inputs are

$$(e_{A1}, e_{A2}, e_{B1}, e_{B2}) = \left(\frac{1+e}{2}, \frac{1-e}{2}, 0, e\right).$$

(c) (5 points) Under what condition on e is there a free trade equilibrium in which country B produces both goods? Compute the equilibrium.

Solution: Country B produces both goods if it is large enough relative to A, so when e is large enough. Zero profit conditions imply

$$p_1 a_{B1} - w_B = 0 \iff w_B = a_{B1} = 1,$$
$$p_2 a_{B2} - w_B = 0 \iff p = w_B / a_{B2} = 1.$$

Since country A has comparative advantage in good 1, A produces only good 1. Hence zero profit condition implies

$$p_1 a_{A1} - w_A = 0 \iff w_A = p a_{A1} = 2.$$

Since country B produces the global supply of good 2, market clearing implies

$$x_2^A + x_2^B = a_{B2} e_{B2} \iff e_{B2} = 1 + \frac{e}{2}.$$

Since by assumption country B produces both goods, by labor market clearing it must be

$$1 + \frac{e}{2} < e \iff e > 2.$$

In this case in equilibrium the price is $(p_1, p_2) = (1, 1)$, consumption is

$$(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = \left(1, 1, \frac{e}{2}, \frac{e}{2}\right),$$

and labor inputs are

$$(e_{A1}, e_{A2}, e_{B1}, e_{B2}) = \left(1, 0, \frac{e}{2} - 1, 1 + \frac{e}{2}\right).$$

(d) (10 points) Under what condition on e is there a free trade equilibrium in which each country produces only one good? Compute the equilibrium.

Solution: In the intermediate case of $1 \le e \le 2$, A specializes in good 1 and B specializes in good 2. Zero profit condition implies

$$p_1 a_{A1} - w_A = 0 \iff w_A = a_{A1} = 2,$$
$$p_2 a_{B2} - w_B = 0 \iff w_B = p a_{B2} = p.$$

Since country A produces the global supply of good 1 and does not produce good 2, market clearing implies

$$x_{A1} + x_{A2} = a_{A1} e_A \iff 1 + \frac{pe}{2} = 2 \iff p = \frac{2}{e}.$$

In this case in equilibrium the price is $(p_1, p_2) = (1, 2/e)$, consumption is

$$(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = \left(1, \frac{e}{2}, 1, \frac{e}{2}\right),$$

and labor inputs are

$$(e_{A1}, e_{A2}, e_{B1}, e_{B2}) = (1, 0, 0, e).$$

4. (30 points) Suppose you are an economist advising a president of some country. Suppose the president would like to impose tariffs against China. Convince the president that that is a bad policy. Assume the president is intelligent and would not be convinced unless you demonstrate using formal mathematical reasoning.

   Solution: You need to argue that whatever trade policy a country adopts, it is always Pareto improving to switch to free trade provided that the government introduces appropriate taxes and subsidies. The formal statements and proofs are in the lecture note.

You can detach this sheet and use as a scratch paper.