Operations Research (B)
Final Exam, Part I

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March 18, 2014

Name: ______________________________

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, basic calculator, straight edge, and your ID card. Failure to do so may be regarded as academic dishonesty.
- The exam time is 180 minutes.
- This exam has two parts, I and II.
- Part I has 11 questions on 9 pages excluding the cover page, for a total of 120 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Submit your entire exam sheet before leaving the room, even if some parts are empty or you intend to drop the class.

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	10	10	10	10	10	10	10	10	20	120
Score:												

1. (a) (2 points) What is your name? Legibly write down your full name as registered to UCSD at the designated area on the cover of *both* exam sheets (Parts I and II).

> **Solution:** This should be easy.

(b) (2 points) Define the vectors a, b by $a = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$. Compute $a + b$.

> **Solution:**
> $$a + b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 1+(-4) \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

(c) (2 points) Define the vectors a, b by $a = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$. Compute the inner product $\langle a, b \rangle$.

> **Solution:**
> $$\langle a, b \rangle = 2 \times 3 + 1 \times (-4) = 6 - 4 = 2.$$

(d) (2 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and vector $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Compute Ab.

> **Solution:**
> $$Ab = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times (-1) \\ 3 \times 1 + 4 \times (-1) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

(e) (2 points) Compute the gradient of $f(x, y) = x + 2y$.

> **Solution:**
> $$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

2. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x > y^2\}$ and $B = \{(x, y) \mid x^2 + (y-3)^2 \leq 5\}$.

(a) (5 points) Draw a picture of the sets A, B on the xy plane.

> **Solution:** A is the inside of a parabola with apex $(0, 0)$ and axis $y = 0$. B is a disk with center $(0, 3)$ and radius $\sqrt{5}$.

(b) (5 points) Can A, B be separated? If so, provide an equation of a straight line that separates them. If not, explain why.

Solution: A, B are nonempty and convex. The point $(1, 1)$ is a common boundary point of A, B. They can be separated by the tangent $y = \frac{1}{2}x + \frac{1}{2}$.

3. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 + 2x, & (x \leq 0) \\ x^2 - 2x. & (x \geq 0) \end{cases}$$

   (a) (3 points) Draw the graph and epigraph of f on the xy plane.

   Solution: Draw two parabolas corresponding to $x \leq 0$ and $x \geq 0$.

   (b) (3 points) Is f convex? Answer yes or no, then explain why.

   Solution: No (1 point), since $(1, -1) \in \operatorname{epi} f$ and $(-1, -1) \in \operatorname{epi} f$ (the apex of each parabola) but $1/2(1, -1) + 1/2(-1, -1) = (0, -1) \notin \operatorname{epi} f$ (2 points).

   (c) (4 points) Let $g = \operatorname{co} f$ be the convex hull of f, so by definition g is a function such that $\operatorname{epi} g = \operatorname{co} \operatorname{epi} f$. Give a formula for $g(x)$.

   Solution:

$$g(x) = \begin{cases} x^2 + 2x, & (x \leq -1) \\ -1, & (-1 \leq x \leq 1) \\ x^2 - 2x. & (x \geq 1) \end{cases}$$

4. There are N types of coins. A coin of type n has integer value v_n. You want to find the minimum number of coins needed for the value of the coins sums to S, where $S \geq 0$ is an integer.

   (a) (2 points) What is (are) the state variable(s)?

   Solution: The state variable is S.

   (b) (4 points) Write down the Bellman equation.

   Solution: Let $V(S)$ be the minimum number of coins the sum of which is S. If you choose type n coin first, you need to sum up to $S - v_n$ with remaining coins. Therefore the Bellman equation is

$$V(S) = \min_n [1 + V(S - v_n)].$$

(c) (4 points) Solve the problem for $S = 10$ when $N = 3$ and $(v_1, v_2, v_3) = (1, 2, 4)$.

> **Solution:** Clearly $V(0) = 0$ and $V(1) = 1$. Iterating the Bellman equation, we get $V(2) = 1$, $V(3) = 2$, $V(4) = 1$, $V(5) = 2$, $V(6) = 2$, $V(7) = 3$, $V(8) = 2$, $V(9) = 3$, $V(10) = 3$. (2 points for correct $V(0)$ and $V(1)$, 2 points for correct $V(10)$.)

5. You have a call option on a stock with strike price K and time to expiration T. This means that if you exercise the option at time $t \leq T$ when the stock price is S_t, you will get $S_t - K$ at t. If you don't exercise the option, you will get nothing. You want to exercise the option so as to maximize the expected discounted payoff

$$\mathrm{E}\left[\frac{1}{(1+r)^t} \max\{S_t - K, 0\}\right],$$

where t is the exercise date and r is the interest rate. Assume that the gross return of the stock is

$$\frac{S_{t+1}}{S_t} = \begin{cases} 1 + \mu + \sigma, & \text{(with probability } \pi_u) \\ 1 + \mu - \sigma, & \text{(with probability } \pi_d) \end{cases}$$

where $\mu > 0$ is the expected return, $\sigma > \mu$ is the volatility, and $\pi_u + \pi_d = 1$.

(a) (2 points) What is (are) the state variable(s)?

> **Solution:** The state variables are the stock price S and the time to expiration T. (1 point each.)

(b) (4 points) Write down the Bellman equation that the option value satisfies.

> **Solution:** Let $V_T(S)$ be the option value when the stock price is S and time to expiration is T. If you exercise, you get $S - K$. If you don't exercise, you go to the next period. Therefore the Bellman equation is
>
> $$V_T(S) = \max\left\{S - K, \frac{1}{1+r}\left(\pi_u V_{T-1}((1+\mu+\sigma)S) + \pi_d V_{T-1}((1+\mu-\sigma)S)\right)\right\}.$$

(c) (4 points) Compute the option value when $T = 1$ and the current stock price is S, where $S < K < (1 + \mu + \sigma)S$.

> **Solution:** Clearly $V_0(S) = \max\{S - K, 0\}$. Since $S < K$, you will not exercise at $t = 0$. Since $\sigma > \mu$ and $(1 + \mu + \sigma)S > K$, you will exercise at $t = 1$ if and only if the stock goes up. Therefore
>
> $$V_1(S) = \frac{1}{1+r}\pi_u((1+\mu+\sigma)S - K).$$

6. You are a potato farmer. You start with some stock of potatoes. At each time, you can eat some of them and plant the rest. If you plant x potatoes, you will harvest Ax^α potatoes at the beginning of the next period, where $A, \alpha > 0$. You want to maximize your utility from consuming potatoes

$$\sum_{t=0}^{T} \beta^t \log c_t,$$

where $0 < \beta < 1$ is the discount factor, $c_t > 0$ is consumption of potatoes at time t, and T is the number of periods you live.

(a) (2 points) If you have k potatoes now and consume c out of it, how many potatoes can you harvest next period?

> **Solution:** If you have k potatoes now and consume c, you will plant $x = k - c$. Therefore you will harvest $Ax^\alpha = A(k-c)^\alpha$ next period.

(b) (3 points) Let $V_T(k)$ be the maximum utility you get when you start with k potatoes. Write down the Bellman equation.

> **Solution:** Let $V_T(k)$ be the maximum utility when starting from stock k and there are T periods to go. If you consume c now, you will start with $A(k-c)^\alpha$ potatoes next period. Therefore the Bellman equation is
>
> $$V_T(k) = \max_{0 \le c \le k} \left[\log c + \beta V_{T-1}(A(k-c)^\alpha)\right].$$

(c) (2 points) Solve for the optimal consumption when $T = 1$.

> **Solution:** Clearly $V_0(k) = \log k$ since you will consume everything if there is no future. By the Bellman equation,
>
> $$V_1(k) = \max_{0 \le c \le k} \left[\log c + \beta \log(A(k-c)^\alpha)\right].$$
>
> The objective function is concave in c. The first-order condition is
>
> $$\frac{1}{c} - \beta\alpha \frac{1}{k-c} = 0 \iff c = \frac{k}{1+\alpha\beta},$$
>
> which is the solution.

(d) (3 points) Guess that $V_T(k) = a_T + b_T \log k$ for some constants a_T, b_T. Assuming that this guess is correct, derive a relation between b_T and b_{T-1}.

Solution: Assuming that the guess is correct, by the Bellman equation we get

$$a_T + b_T \log k = \max_{0 \le c \le k} \left[\log c + \beta(a_{T-1} + b_{T-1} \log(A(k-c)^\alpha)) \right].$$

By the first-order condition we get

$$\frac{1}{c} - \beta\alpha b_{T-1} \frac{1}{k-c} = 0 \iff c = \frac{k}{1 + \alpha\beta b_{T-1}}.$$

Substituting the optimal consumption into the Bellman equation and comparing the coefficients of $\log k$, we get

$$b_T = 1 + \alpha\beta b_{T-1}.$$

7. Consider the problem

$$\begin{array}{ll} \text{minimize} & 3x_1 + x_2 \\ \text{subject to} & x_2 \le -x_1^2, \\ & x_1^2 + (x_2 - 1)^2 \le 1. \end{array}$$

(a) (2 points) Draw a picture of the set that each constraint defines (in one picture).

Solution: The constraint $x_2 \le -x_1^2$ is a hyperbola with apex $(0, 0)$ (1 point). The constraint $x_1^2 + (x_2 - 1)^2 \le 1$ is a disk with center $(0, 1)$ and radius 1 (1 point).

(b) (2 points) Compute the solution.

Solution: Since the constraint consists of the single point $(x_1, x_2) = (0, 0)$, it is the unique solution.

(c) (3 points) Compute the tangent cone and the linearizing cone at the solution.

Solution: Let $\bar{x} = (x_1, x_2) = (0, 0)$. Since the constraint set is $C = \{\bar{x}\}$, we have $\bar{x} + ty \in C$ with $t > 0$ only if $y = (0, 0)$. Therefore the tangent cone is $T(\bar{x}) = \{0\}$ (1 point).
Let $g_1(x) = x_1^2 + x_2$ and $g_2(x) = x_1^2 + (x_2 - 1)^2 - 1$. Then

$$\nabla g_1(\bar{x}) = \begin{bmatrix} 2x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \nabla g_2(\bar{x}) = \begin{bmatrix} 2x_1 \\ 2(x_2 - 1) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix},$$

so the linearizing cone is $L(\bar{x}) = \{(y_1, y_2) \mid y_2 = 0\}$ (2 points).

(d) (3 points) Do the Karush-Kuhn-Tucker conditions hold? If so, give the Lagrange multipliers. If not, explain why.

> **Solution:** Let
>
> $$L(x_1, x_2, \lambda_1, \lambda_2) = 3x_1 + x_2 + \lambda_1(x_1^2 + x_2) + \lambda_2(x_1^2 + (x_2 - 1)^2 - 1)$$
>
> be the Lagrangian. If the KKT theorem holds, we must have
>
> $$0 = \frac{\partial L}{\partial x_1} = 3 + 2\lambda_1 x_1 + 2\lambda_2 x_1,$$
> $$0 = \frac{\partial L}{\partial x_2} = 1 + \lambda_1 + 2\lambda_2(x_2 - 1)$$
>
> at $\bar{x} = (x_1, x_2) = (0, 0)$, but the first equation becomes $0 = 3$, a contradiction. Therefore the KKT conditions do not hold (1 point).
>
> The reason is because the Guignard constraint qualification $L(\bar{x}) \subset \operatorname{co} T(\bar{x})$ does not hold (2 points).

8. Consider the problem

$$\begin{aligned} &\text{maximize} && x_1^2 + 2x_2^2 \\ &\text{subject to} && x_1 + x_2 \le 1, \\ &&& x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

(a) (2 points) Are the Karush-Kuhn-Tucker conditions necessary for a solution? Answer yes or no, then explain why.

> **Solution:** Yes (1 point). Since the objective function is continuous and the constraint set is compact, there is a solution. Since the constraints are linear, the constraint qualification holds automatically. Therefore the KKT conditions hold (1 point).

(b) (2 points) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? Answer yes or no, then explain why.

> **Solution:** No (1 point). Since the objective function is convex but the problem is a maximization problem, the first-order conditions are not sufficient (1 point). Note that KKT conditions are sufficient for *convex minimization* or *concave maximization* problems.

(c) (2 points) Write down the Lagrangian.

Solution:

$$L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = x_1^2 + 2x_2^2 + \lambda_1(1 - x_1 - x_2) + \lambda_2 x_1 + \lambda_3 x_2.$$

(d) (4 points) Compute the solution.

Solution: Since the objective function is increasing in both x_1 and x_2, clearly the constraint $x_1 + x_2 \leq 1$ binds. Therefore $x_1 + x_2 = 1$. The first-order condition is

$$0 = \frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 + \lambda_2,$$
$$0 = \frac{\partial L}{\partial x_2} = 4x_2 - \lambda_1 + \lambda_3.$$

If $x_1 = 0$, then $x_2 = 1$, and the function value is 2. If $x_2 = 0$, then $x_1 = 1$, and the function value is 1. If $x_1, x_2 > 0$, then by complementary slackness $\lambda_2 = \lambda_3 = 0$. Solving the first-order conditions we get $x_1 = \lambda_1/2$ and $x_2 = \lambda_1/4$. Since $x_1 + x_2 = 1$, we get $x_1 = \frac{2}{3}$ and $x_2 = \frac{1}{3}$. Then the function value is

$$\left(\frac{2}{3}\right)^2 + 2\left(\frac{1}{3}\right)^2 = \frac{4+2}{9} = \frac{2}{3}.$$

Therefore the solution is $(x_1, x_2) = (0, 1)$.

9. Consider the problem

$$\begin{array}{ll} \text{maximize} & \log x_1 + u_1 \log x_2 \\ \text{subject to} & x_1 + x_2 \leq u_2, \end{array}$$

where $x_1, x_2 > 0$ are variables and $u_1, u_2 > 0$ are parameters.

(a) (2 points) Prove that the objective function is concave. (Hint: by definition f is concave if $-f$ is convex.)

Solution: Since $(\log x)'' = (1/x)' = -1/x^2 < 0$, $f(x) = \log x$ is concave. Since the objective function is $f(x_1) + u_1 f(x_2)$, it is concave.

(b) (1 point) Write down the Lagrangian.

Solution:

$$L(x_1, x_2, \lambda, u_1, u_2) = \log x_1 + u_1 \log x_2 + \lambda(u_2 - x_1 - x_2).$$

(c) (3 points) Compute the solution.

Solution: Since the objective function is concave, the constraint is convex, and the Slater condition holds, the KKT conditions are necessary and sufficient for a solution. The first-order condition is

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} = \frac{1}{x_1} - \lambda \iff x_1 = \frac{1}{\lambda}, \\ 0 &= \frac{\partial L}{\partial x_2} = \frac{u_1}{x_2} - \lambda \iff x_2 = \frac{u_1}{\lambda}. \end{aligned}$$

Clearly $\lambda > 0$. By complementary slackness, we get $x_1 + x_2 = u_2$, so $\lambda = \frac{1+u_1}{u_2}$ and the solution is

$$(x_1, x_2) = \left(\frac{u_2}{1+u_1}, \frac{u_1 u_2}{1+u_1} \right).$$

(d) (4 points) Let $\phi(u_1, u_2)$ be the maximum value of the problem. Compute $\frac{\partial \phi}{\partial u_1}$ and $\frac{\partial \phi}{\partial u_2}$.

Solution: Let $u = (u_1, u_2)$. By the envelope theorem,

$$\begin{aligned} \begin{bmatrix} \frac{\partial \phi}{\partial u_1} \\ \frac{\partial \phi}{\partial u_2} \end{bmatrix} &= \nabla_u \phi(u) = \nabla_u L(x(u), \lambda(u), u) \\ &= \begin{bmatrix} \log x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} \log \frac{u_1 u_2}{1+u_1} \\ \frac{1+u_1}{u_2} \end{bmatrix}. \end{aligned}$$

Alternatively, you can compute $\phi(u_1, u_2)$ and its partial derivatives.

10. Consider the problem

$$\begin{aligned} &\text{minimize} && \sum_{n=1}^{N} [x_n \log x_n - x_n] \\ &\text{subject to} && \sum_{n=1}^{N} a_n x_n \le b, \end{aligned}$$

where $x_n > 0$ are variables and a_n, b are constants.

(a) (2 points) Write down the Lagrangian $L(x, \lambda)$, where $x = (x_1, \dots, x_N)$ and $\lambda \ge 0$ is the Lagrange multiplier.

Solution:

$$L(x, \lambda) = \sum_{n=1}^{N} [x_n \log x_n - x_n] + \lambda \left(\sum_{n=1}^{N} a_n x_n - b \right).$$

(b) (3 points) Minimize the Lagrangian with respect to $x = (x_1, \ldots, x_N)$ and express x_n as a function of λ.

> **Solution:** L is convex in x. The first-order condition is
>
> $$0 = \frac{\partial L}{\partial x_n} = \log x_n + \lambda a_n \iff x_n = \mathrm{e}^{-a_n \lambda}.$$

(c) (5 points) Derive the dual problem.

> **Solution:** Substituting $x_n = \mathrm{e}^{-a_n \lambda}$, the dual objective function is
>
> $$\omega(\lambda) = \min_x L(x, \lambda) = -b\lambda - \sum_{n=1}^{N} \mathrm{e}^{-a_n \lambda}.$$
>
> Therefore the dual problem is
>
> $$\begin{aligned} &\text{maximize} && -b\lambda - \sum_{n=1}^{N} \mathrm{e}^{-a_n \lambda} \\ &\text{subject to} && \lambda \geq 0. \end{aligned}$$

11. (Extra credit)

(a) (5 points) State the Separating Hyperplane Theorem. (There are a few versions, but any one of them is fine. However, the statement must be mathematically precise.)

> **Solution:** Let $C, D \subset \mathbb{R}^N$ be nonempty convex sets. If $C \cap D = \emptyset$, then there exists a nonzero vector $a \in \mathbb{R}^N$ such that $\langle a, x \rangle \leq \langle a, y \rangle$ for all $x \in C$ and $y \in D$.

(b) (15 points) State and prove your favorite theorem that uses the Separating Hyperplane Theorem in the proof.