Operations Research (B)
Midterm Exam 1

Prof. Alexis Akira Toda

January 23, 2014

Name: ____

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, basic calculator, straight edge, and your ID card. Failure to do so may be regarded as academic dishonesty.
- The exam time is 80 minutes.
- This exam has 7 questions on 3 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Once you exit the exam room, you are not allowed to reenter.
- Submit your entire exam sheet before leaving the room, even if some parts are empty or you intend to drop the class.

Question:	1	2	3	4	5	6	7	Total
Points:	5	20	15	15	15	15	15	100
Score:								

1. (5 points) What is your name? Legibly write down your full name as registered to UCSD at the designated area on the cover of this exam sheet.

Solution: This should be easy.

2. (a) (5 points) Compute the inner product of $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Solution: $\langle x, y \rangle = 1 \times 3 + 2 \times 4 = 11.$

(b) (5 points) Let A, B be two square matrices of the same size. Is necessarily $AB = BA$? If so, prove it. If not, provide a counterexample.

Solution: No (2 points). Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

so $AB \neq BA$ (3 points for a counterexample).

(c) (5 points) What is the definition of a convex function? (Several definitions are possible; pick your favorite one.)

Solution: f is convex if for any vectors x, y and $0 \leq \alpha \leq 1$, we have

$$f((1 - \alpha)x + \alpha y) \leq (1 - \alpha) f(x) + \alpha f(y).$$

Alternatively, f is a convex function if its epigraph $\operatorname{epi} f = \{(x, y) \,|\, y \geq f(x)\}$ is a convex set.

($f''(x) \geq 0$ is *not* the definition of a convex function. First, f may not be differentiable. Second, x can a vector. However, it is true that if $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, then f is convex if and only if $f''(x) \geq 0$.)

(d) (5 points) What is the definition of a convex hull?

Solution: The convex hull of a set A is the smallest convex set that contains A.

3. Let $A = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq x^4\}$.

(a) (5 points) Draw a picture of the set A on the xy plane.

Solution: This should be easy.

(b) (5 points) Is A convex? Answer yes or no, then explain why.

Solution: Yes (2 points), because A is the epigraph of the convex function $f(x) = x^4$ (3 points). (f is convex because $f''(x) = 12x^2 \geq 0$.)

(c) (5 points) What is the convex hull of A? Answer as "$\operatorname{co} A = \{(x, y) \in \mathbb{R}^2 \mid \text{blabla}\}$".

Solution: $\operatorname{co} A = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^4\} = A$.

4. Let $B = \{(x, y) \in \mathbb{R}^2 \mid y > x^3\}$ and $C = \{(x, y) \in \mathbb{R}^2 \mid x \geq 1, y \leq 1\}$.

(a) (5 points) Draw a picture of the sets B, C on the xy plane.

Solution: This should be easy.

(b) (10 points) Can B, C be separated? If so, provide an equation of a straight line that separates them. If not, explain why.

Solution: No (5 points). This is because if B, C can be separated, so must be $\operatorname{co} B$ and C, but $\operatorname{co} B = \mathbb{R}^2$ intersects C (5 points). (Alternatively, if there is a separating line it has to be the tangent of B at $(1, 1)$, which is $y = 3x - 2$, but this line intersects B since $x^3 < 3x - 2$ as $x \to -\infty$.)

5. Let $D = \{(x, y) \in \mathbb{R}^2 \mid y < \log x\}$ and $E = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0\}$.

(a) (5 points) Draw a picture of the sets D, E on the xy plane.

Solution: This should be easy.

(b) (5 points) Provide an equation of a straight line that separates D, E.

Solution: $x = 0$.

(c) (5 points) Can D, E be strictly separated? Answer yes or no, then explain why.

Solution: No (2 points). Since E contains vertical lines, if D, E can be separated it has to be a vertical line. Since $(x, y) \in D$ if $y < \log x$ and $x > 0$ can be arbitrarily small by letting $y \to -\infty$, the separating line must be $x = 0$. But this line is contained in E, so D, E cannot be strictly separated (3 points).

6. (15 points) Let $F = \{(x, y) \in \mathbb{R}^2 \mid y \geq e^x\}$ and P be the point with coordinates $(3, 2)$. Let (a, e^a) be the nearest point in F to P. Derive an equation that a satisfies.

Solution: Since $(e^x)' = e^x$, the tangent vector to F at (a, e^a) is $\begin{bmatrix} 1 \\ e^a \end{bmatrix}$. In order for P to be nearest to F, the tangent vector must make a right angle with the vector from $(3, 2)$ to (a, e^a) (8 points). Therefore the desired equation is

$$\begin{bmatrix} 1 \\ e^a \end{bmatrix} \cdot \begin{bmatrix} a-3 \\ e^a - 2 \end{bmatrix} = 0 \iff a - 3 + e^{2a} - 2e^a = 0.$$

(Alternatively, you can take the first-order condition of minimizing $(a-3)^2 + (e^a - 2)^2$.)

7. Let $a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

   (a) (5 points) Draw a picture of $C = \text{cone}[a_1, a_2]$, the cone generated by a_1, a_2 on the xy plane.

   Solution: This should be easy.

   (b) (10 points) Draw a picture of the dual cone C^* on the same plane as part (a). Clearly show the boundary of C^*.

   Solution: The vectors orthogonal to a_1, a_2 are $b_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ (5 points), so C^* is the cone generated by b_1, b_2 (5 points). (Draw a picture to see for yourself.)