Operations Research (B)
Midterm Exam 2

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Name: ____

Instruction:

- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, basic calculator, straight edge, and your ID card. Failure to do so may be regarded as academic dishonesty.
- The exam time is 80 minutes.
- This exam has 6 questions on 6 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use the back of the exam sheet, but make sure to indicate that you are using the back.
- Once you exit the exam room, you are not allowed to reenter.
- Submit your entire exam sheet before leaving the room, even if some parts are empty or you intend to drop the class.

Question:	1	2	3	4	5	6	Total
Points:	10	10	20	20	20	20	100
Score:							

1. (a) (5 points) What is your name? Legibly write down your full name as registered to UCSD at the designated area on the cover of this exam sheet.

   Solution: This should be easy.

   (b) (5 points) Compute the gradient of $f(x, y) = 2x^2 + 3xy + 5y^2$.

   Solution:

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 4x + 3y \\ 3y + 10y \end{bmatrix}.$$

   (3 points for knowing the definition of the gradient, 2 points for the correct answer.)

2. Consider the problem

$$\begin{array}{ll} \text{minimize} & (x_1 - 2)^2 + (x_2 - 3)^2 \\ \text{subject to} & (x_1 - 1)^2 + x_2^2 \leq 5, \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

   (a) (5 points) Are the Karush-Kuhn-Tucker conditions necessary for a solution? That is, does the solution to the problem have to satisfy the KKT conditions? Answer yes or no, then explain why.

   Solution: Yes (2 points). Since the objective function is convex, the constraints are convex, and the Slater constraint qualification holds for $(x_1, x_2) = (1, \epsilon)$ with $\epsilon > 0$ small enough, the KKT theorem applies (3 points).

   (b) (5 points) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? That is, does a point satisfying the KKT conditions have to be a solution to the problem? Answer yes or no, then explain why.

   Solution: Yes (2 points). Since the objective function and constraints are convex, the KKT conditions are sufficient for optimality (3 points).

3. Consider the problem

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 \\ \text{subject to} & x_1 \geq 2, \\ & x_1^2 + x_2^2 \leq 4. \end{array}$$

   (a) (5 points) Draw a picture of the set that each constraint defines.

   Solution: The constraint $x_1 \geq 2$ is a half space (2 points). The constraint $x_1^2 + x_2^2 \leq 4$ is a disk with center $(0, 0)$ and radius 2 (3 points).

(b) (5 points) Compute the solution.

Solution: Since the constraint consists of the single point $(x_1, x_2) = (2, 0)$, it is the unique solution.

(c) (5 points) Compute the tangent cone and the linearizing cone at the solution.

Solution: Let $\bar{x} = (x_1, x_2) = (2, 0)$. Since the constraint set is $C = \{\bar{x}\}$, we have $\bar{x} + ty \in C$ with $t > 0$ only if $y = (0, 0)$. Therefore the tangent cone is $T(\bar{x}) = \{0\}$ (2 points).
Let $g_1(x) = 2 - x_1$ and $g_2(x) = x_1^2 + x_2^2 - 4$. Then

$$\nabla g_1(\bar{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ \nabla g_2(\bar{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix},$$

so the linearizing cone is $L(\bar{x}) = \{(y_1, y_2) \mid y_1 = 0\}$ (3 points).

(d) (5 points) Do the Karush-Kuhn-Tucker conditions hold? If so, give the Lagrange multipliers. If not, explain why.

Solution: Let

$$L(x_1, x_2, \lambda_1, \lambda_2) = 2x_1 + x_2 + \lambda_1(2 - x_1) + \lambda_2(x_1^2 + x_2^2 - 4)$$

be the Lagrangian. If the KKT theorem holds, we must have

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} = 2 - \lambda_1 + 2\lambda_2 x_1, \\ 0 &= \frac{\partial L}{\partial x_2} = 1 + 2\lambda_2 x_2 \end{aligned}$$

at $\bar{x} = (x_1, x_2) = (2, 0)$, but the second equation becomes $0 = 1$, a contradiction. Therefore the KKT conditions do not hold (2 points).
The reason is because the Guignard constraint qualification $L(\bar{x}) \subset \operatorname{co} T(\bar{x})$ does not hold (3 points).

4. Consider the problem

$$\begin{array}{ll} \text{minimize} & \dfrac{1}{x_1} + \dfrac{4}{x_2} \\ \text{subject to} & x_1 + x_2 \le 3, \end{array}$$

where $x_1, x_2 > 0$.

(a) (5 points) Prove that the objective function is convex.

Solution: Since $(1/x)'' = (-1/x^2)' = 2/x^3 > 0$, both $1/x_1$ and $4/x_2$ are convex. Therefore $1/x_1 + 4/x_2$ is convex.

(b) (5 points) Write down the Lagrangian.

Solution:

$$L(x_1, x_2, \lambda) = \frac{1}{x_1} + \frac{4}{x_2} + \lambda(x_1 + x_2 - 3).$$

(c) (5 points) Explain why the Karush-Kuhn-Tucker conditions are both necessary and sufficient for a solution.

Solution: Since the objective function and constraints are convex and the Slater condition holds for $(x_1, x_2) = (1, 1)$, the KKT conditions are both necessary and sufficient. (2 points for stating convexity, 2 points for stating SCQ, and 1 point for giving a point satisfying SCQ.)

(d) (5 points) Compute the solution.

Solution: The first-order condition is

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} = -\frac{1}{x_1^2} + \lambda \iff x_1 = \frac{1}{\sqrt{\lambda}}, \\ 0 &= \frac{\partial L}{\partial x_2} = -\frac{4}{x_1^2} + \lambda \iff x_2 = \frac{2}{\sqrt{\lambda}}. \end{aligned}$$

Clearly $\lambda > 0$. The complementary slackness condition is then

$$\lambda(x_1 + x_2 - 3) = 0 \iff \frac{3}{\sqrt{\lambda}} = 3 \iff \lambda = 1,$$

so the solution is $(x_1, x_2) = (1, 2)$. (2 points for first-order condition, 1 point for complementary slackness, 2 points for solution.)

5. Consider the problem

$$\begin{aligned} &\text{maximize} && \frac{4}{3}x_1^3 + \frac{1}{3}x_2^3 \\ &\text{subject to} && x_1 + x_2 \le 1, \\ & && x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

(a) (5 points) Are the Karush-Kuhn-Tucker conditions necessary for a solution? Answer yes or no, then explain why.

Solution: Yes (2 points). Since the objective function is continuous and the constraint set is compact, there is a solution. Since the constraints are linear, the constraint qualification holds automatically. Therefore the KKT conditions hold (3 points).

(b) (5 points) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? Answer yes or no, then explain why.

Solution: No (2 points). Since the objective function is convex but the problem is a maximization problem, the first-order conditions are not sufficient (3 points). Note that KKT conditions are sufficient for *convex minimization* or *concave maximization* problems.

(c) (5 points) Write down the Lagrangian.

Solution:

$$L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = \frac{4}{3}x_1^3 + \frac{1}{3}x_2^3 + \lambda_1(1 - x_1 - x_2) + \lambda_2 x_1 + \lambda_3 x_2.$$

(d) (5 points) Compute the solution.

Solution: Since the objective function is increasing in both x_1 and x_2, clearly the constraint $x_1 + x_2 \leq 1$ binds. Therefore $x_1 + x_2 = 1$. The first-order condition is

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} = 4x_1^2 - \lambda_1 + \lambda_2, \\ 0 &= \frac{\partial L}{\partial x_2} = x_2^2 - \lambda_1 + \lambda_3. \end{aligned}$$

If $x_1 = 0$, then $x_2 = 1$, and the function value is $\frac{1}{3}$. If $x_2 = 0$, then $x_1 = 1$, and the function value is $\frac{4}{3}$. If $x_1, x_2 > 0$, then by complementary slackness $\lambda_2 = \lambda_3 = 0$. Solving the first-order conditions we get $x_1 = \frac{1}{2\sqrt{\lambda_1}}$ and $x_2 = \frac{1}{\sqrt{\lambda_1}}$. Since $x_1 + x_2 = 1$, we get $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$. Then the function value is

$$\frac{4}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{3}\left(\frac{2}{3}\right)^3 = \frac{4+8}{81} = \frac{4}{27}.$$

Therefore the solution is $(x_1, x_2) = (1, 0)$.

6. Consider the problem

$$\begin{array}{ll} \text{maximize} & \sqrt{x_1} + u_1\sqrt{x_2} \\ \text{subject to} & x_1 + x_2 \le u_2, \end{array}$$

where $x_1, x_2 > 0$ are variables and $u_1, u_2 > 0$ are parameters.

(a) (5 points) Prove that the objective function is concave. (Hint: by definition f is concave if $-f$ is convex.)

> **Solution:** Since $(\sqrt{x})'' = (\frac{1}{2}x^{-\frac{1}{2}})' = -\frac{1}{4}x^{-\frac{3}{2}} < 0$, $f(x) = \sqrt{x}$ is concave. Since the objective function is $f(x_1) + u_1 f(x_2)$, it is concave.

(b) (5 points) Write down the Lagrangian.

> **Solution:**
>
> $$L(x_1, x_2, \lambda, u_1, u_2) = \sqrt{x_1} + u_1\sqrt{x_2} + \lambda(u_2 - x_1 - x_2).$$

(c) (5 points) Compute the solution.

> **Solution:** Since the objective function is concave, the constraint is convex, and the Slater condition holds, the KKT conditions are necessary and sufficient for a solution. The first-order condition is
>
> $$0 = \frac{\partial L}{\partial x_1} = \frac{1}{2\sqrt{x_1}} - \lambda \iff x_1 = \frac{1}{4\lambda^2},$$
> $$0 = \frac{\partial L}{\partial x_2} = \frac{u_1}{2\sqrt{x_2}} - \lambda \iff x_2 = \frac{u_1^2}{4\lambda^2}.$$
>
> Clearly $\lambda > 0$. By complementary slackness, we get $x_1 + x_2 = u_2$, so $\lambda = \frac{\sqrt{1+u_1^2}}{2\sqrt{u_2}}$ and the solution is
>
> $$(x_1, x_2) = \left(\frac{u_2}{1+u_1^2}, \frac{u_1^2 u_2}{1+u_1^2}\right).$$
>
> (2 points for first-order condition, 1 point for complementary slackness, 2 points for solution.)

(d) (5 points) Let $\phi(u_1, u_2)$ be the maximum value of the problem. Compute $\frac{\partial \phi}{\partial u_1}$ and $\frac{\partial \phi}{\partial u_2}$.

Solution: Let $u = (u_1, u_2)$. By the envelope theorem,

$$\begin{bmatrix} \frac{\partial \phi}{\partial u_1} \\ \frac{\partial \phi}{\partial u_2} \end{bmatrix} = \nabla_u \phi(u) = \nabla_u L(x(u), \lambda(u), u)$$
$$= \begin{bmatrix} \sqrt{x_2} \\ \lambda \end{bmatrix} = \begin{bmatrix} \frac{u_1 \sqrt{u_2}}{\sqrt{1+u_1^2}} \\ \frac{\sqrt{1+u_1^2}}{2\sqrt{u_2}} \end{bmatrix}.$$

Alternatively, you can compute $\phi(u_1, u_2)$ and its partial derivatives.