Operations Research (B)
Final Exam

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Name: ______________________________

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2-2x+1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 180 minutes.
- This exam has 9 questions on 11 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	15	10	10	10	13	10	15	7	10	100
Score:										

1. Let $f(x) = \sqrt{x^2+4}$.

(a) (2 points) Compute the first and second derivatives of f.

Solution: By the chain rule and the quotient rule, we get

$$f'(x) = \frac{1}{2}\frac{2x}{\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}},$$

$$f''(x) = \frac{\sqrt{x^2+4} - x\frac{x}{\sqrt{x^2+4}}}{x^2+4} = \frac{4}{(x^2+4)^{\frac{3}{2}}}.$$

(b) (2 points) Is f convex, concave, or neither?

Solution: Since $f''(x) = \frac{4}{(x^2+4)^{\frac{3}{2}}} \geq 0$, it is convex.

(c) (2 points) Find the value of x that minimizes $f(x)$.

Solution: Since $0 = f'(x) = \frac{x}{\sqrt{x^2+4}} \iff x = 0$, the solution is $x = 0$.

(d) (2 points) Suppose you want to minimize f by using the Newton algorithm. Let x_n be the approximate solution at iteration n. Express x_{n+1} using **only** x_n.

Solution: By the definition of the Newton algorithm,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f'(x_n)}{f''(x_n)} \\ &= x_n - \frac{x_n}{\sqrt{x_n^2+4}}\frac{(x_n^2+4)^{\frac{3}{2}}}{4} \\ &= x_n - \frac{1}{4}x_n(x_n^2+4) = -\frac{1}{4}x_n^3. \end{aligned}$$

(e) (2 points) What is the order of convergence of the Newton algorithm for this particular example?

Solution: Recall that if x^* is the limit and there exist numbers $\alpha \geq 1$ and $\beta > 0$ such that

$$|x_{n+1} - x^*| \leq \beta \left|x_n - x^*\right|^{\alpha},$$

then the order of convergence is α. For this example, since $|x_{n+1}| = \frac{1}{4}\left|x_n\right|^3$, the order of convergence is 3 (by setting $x^* = 0$, $\alpha = 3$, and $\beta = \frac{1}{4}$).

(f) (5 points) Let x_0 be the initial value for the Newton algorithm. Prove that if $|x_0| \geq 2$, then the Newton algorithm does not converge to the solution.

Solution: Since $|x_{n+1}| = \frac{1}{4}|x_n|^3$, if $|x_0| \geq 2$, then by induction we get $|x_n| \geq 2$ for all n, so $\{x_n\}$ does not converge to 0.

2. Let $f(x_1, x_2) = x_1^2 - 2x_1x_2 + 2x_2^2 - 4x_1 + 2x_2$.

   (a) (3 points) Compute the gradient and the Hessian of f.

Solution:

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 - 2x_2 - 4 \\ -2x_1 + 4x_2 + 2 \end{bmatrix},$$
$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}.$$

   (b) (2 points) Determine whether f is convex, concave, or neither.

Solution: Since $2 > 0$ and $2 \times 4 - (-2)^2 = 4 > 0$, the Hessian is positive definite. Therefore f is convex.

   (c) (3 points) Find the stationary point(s) of f.

Solution: By the first-order condition, we get

$$\nabla f(x_1, x_2) = 0 \iff (x_1, x_2) = (3, 1),$$

so this is the only stationary point.

   (d) (2 points) Determine whether each stationary point is a maximum, minimum, or neither.

Solution: Since f is convex, a stationary point is the minimum. Therefore $(x_1, x_2) = (3, 1)$ is the minimum of f.

3. Let $f(x_1, x_2) = x_1^3 + 3x_1^2 + x_1x_2 + x_2^2 - 5x_2 + 6$.

   (a) (2 points) Compute the gradient and the Hessian of f.

Solution:

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 + 6x_1 + x_2 \\ x_1 + 2x_2 - 5 \end{bmatrix},$$
$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 + 6 & 1 \\ 1 & 2 \end{bmatrix}.$$

(2 points for the gradient, 1 point for the Hessian.)

(b) (4 points) Find the stationary point(s) of f.

Solution: By the first-order condition, we get

$$\nabla f(x_1, x_2) = 0 \iff \begin{bmatrix} 3x_1^2 + 6x_1 + x_2 \\ x_1 + 2x_2 - 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From the first equation, we get $x_2 = -3x_1^2 - 6x_1$. Substituting into the second equation, we get

$$\begin{aligned} x_1 + 2(-3x_1^2 - 6x_1) - 5 = 0 &\iff 6x_1^2 + 11x_1 + 5 = 0 \\ &\iff (6x_1 + 5)(x_1 + 1) = 0 \\ &\iff x_1 = -1, -\frac{5}{6}. \end{aligned}$$

If $x_1 = -1$, then $x_2 = 3$. If $x_1 = -\frac{5}{6}$, then $x_2 = \frac{35}{12}$. Therefore the stationary points are

$$(x_1, x_2) = (-1, 3), \left(-\frac{5}{6}, \frac{35}{12}\right).$$

(c) (4 points) Determine whether each stationary point is a local maximum, local minimum, or a saddle point.

Solution: At $x_1 = -1$, the Hessian is

$$H = \nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Since the determinant is $0 \times 2 - 1^2 = -1 < 0$, it is a saddle point.
At $x_1 = -\frac{5}{6}$, the Hessian is

$$H = \nabla^2 f(x_1, x_2) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Since $1 > 0$ and $1 \times 2 - 1^2 = 1 > 0$, it is positive definite. Therefore it is a local minimum.

4. Consider the problem

$$\begin{array}{ll} \text{minimize} & x_1 + 6x_2 \\ \text{subject to} & x_1 + x_2^2 \le 0, \\ & x_1 + 2x_2 \ge 1. \end{array}$$

(a) (2 points) Draw a picture of the set that each constraint defines on the x_1x_2 plane.

> **Solution:** The constraint $x_1 \le -x_2^2$ is a parabola with apex $(0,0)$ and axis $x_2 = 0$ (1 point). The constraint $x_1 + 2x_2 \ge 1$ is a straight line (1 point). The boundaries are tangent at the point $(x_1, x_2) = (-1, 1)$.

(b) (2 points) Compute the solution.

> **Solution:** Since the constraint set consists of the single point $(x_1, x_2) = (-1, 1)$, it is the unique solution.

(c) (3 points) Do the Karush-Kuhn-Tucker conditions hold? If so, give the Lagrange multipliers. If not, prove it.

> **Solution:** Let
>
> $$L(x_1, x_2, \lambda_1, \lambda_2) = x_1 + 6x_2 + \lambda_1(x_1 + x_2^2) + \lambda_2(1 - x_1 - 2x_2)$$
>
> be the Lagrangian. If the KKT theorem holds, we must have
>
> $$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} = 1 + \lambda_1 - \lambda_2, \\ 0 &= \frac{\partial L}{\partial x_2} = 6 + 2\lambda_1 x_2 - 2\lambda_2 \end{aligned}$$
>
> at $(x_1, x_2) = (-1, 1)$. Subtracting 2 times the first equation from the second equation, we get $0 = 4$, which is a contradiction. Therefore the KKT conditions do not hold.

(d) (3 points) Explain why the KKT conditions do or do not hold in the previous question.

> **Solution:** It is because the Slater condition fails. Indeed, if the Slater condition holds, then there exist (x_1, x_2) such that $x_1 < -x_2^2$ and $x_1 > 1 - 2x_2$. Combining these two inequalities, we get
>
> $$-x_2^2 > 1 - 2x_2 \iff x_2^2 - 2x_2 + 1 < 0 \iff (x_2 - 1)^2 < 0,$$
>
> which is a contradiction.

5. Consider the problem

$$\begin{aligned} \text{maximize} \quad & \sqrt{x_1} + 2\sqrt{x_2} \\ \text{subject to} \quad & x_1 + x_2 \le 1, \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

(a) (3 points) Are the Karush-Kuhn-Tucker conditions necessary for a solution? Answer yes or no, then explain why.

> **Solution:** Yes (2 points). Since the objective function is continuous and the constraint set is compact, there is a solution. Since the constraints are linear, the constraint qualification holds automatically. Therefore the KKT conditions hold (1 point).

(b) (3 points) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? Answer yes or no, then explain why.

> **Solution:** Yes (2 points). Since the objective function is concave and it is a maximization problem, the first-order conditions are sufficient (1 point).

(c) (3 points) Write down the Lagrangian.

> **Solution:**
>
> $$L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = \sqrt{x_1} + 2\sqrt{x_2} + \lambda_1(1 - x_1 - x_2) + \lambda_2 x_1 + \lambda_3 x_2.$$

(d) (4 points) Compute the solution.

> **Solution:** The first-order conditions are
>
> $$0 = \frac{\partial L}{\partial x_1} = \frac{1}{2\sqrt{x_1}} - \lambda_1 + \lambda_2,$$
> $$0 = \frac{\partial L}{\partial x_1} = \frac{1}{\sqrt{x_2}} - \lambda_1 + \lambda_3.$$
>
> From these equations, it must be $x_1, x_2 > 0$ because otherwise there will be division by 0, which is nonsense. Therefore by complementary slackness $\lambda_2 = \lambda_3 = 0$. Then $x_1 = \frac{1}{4\lambda_1^2}$ and $x_2 = \frac{1}{\lambda_1^2}$, so $\lambda_1 > 0$. Again by complementary slackness, we get
>
> $$0 = 1 - x_1 - x_2 = 1 - \frac{5}{4\lambda^2} \iff \frac{1}{\lambda^2} = \frac{4}{5}.$$
>
> Therefore the solution is $x_1 = \frac{1}{5}$ and $x_2 = \frac{4}{5}$.

6. Suppose that there are two firms, 1 and 2, each producing a single good. Let x_1, x_2 be the output of the good produced by firm 1 and 2, respectively. If the total output is $x = x_1 + x_2$, then the price of the good will be $p(x) = a - bx$, where $a, b > 0$. (If $x > a/b$, then the price is zero.) For simplicity, the cost for producing the good is zero.

The timing is as follows. First, firm 1 decides how much to produce. Second, observing firm 1's output, firm 2 produces the good. The goal of each firm is to maximize their own profit (which equals revenue, because the cost is zero).

(a) (2 points) What are the state and control variables of firm 2's problem?

> **Solution:** The state variable is x_1, the output of firm 1. The control variable is x_2, the output of firm 2.

(b) (2 points) If firm 1 produces x_1, firm 2 produces x_2, and $x_1 + x_2 \leq a/b$, what is the profit of firm 1 and 2?

> **Solution:** Since the price is $p = a - b(x_1 + x_2)$, the profits are
>
> $$\pi_1(x_1, x_2) = (a - b(x_1 + x_2))x_1,$$
> $$\pi_2(x_1, x_2) = (a - b(x_1 + x_2))x_2.$$

(c) (2 points) Write down the Bellman equation of firm 2's problem.

> **Solution:** Suppose firm 1 produces $0 \leq x_1 \leq a/b$. Then the value function of firm 2 is
>
> $$V(x_1) = \max_{x_2} \pi_2(x_1, x_2) = \max_{x_2}(a - b(x_1 + x_2))x_2.$$

(d) (2 points) Solve for the optimal production plan of firm 2, given firm 1's decision.

> **Solution:** The objective function is a concave quadratic function of x_2. By the first-order condition, the solution is
>
> $$a - bx_1 - 2bx_2 = 0 \iff x_2 = \frac{a - bx_1}{2b}.$$

(e) (2 points) Compute the optimal production plan of firm 1 and 2.

> **Solution:** If firm 1 produces x_1, then firm 2 produces $x_2 = \frac{a-bx_1}{2b}$. Then firm 1's profit is
>
> $$\pi_1(x_1, x_2) = (a - b(x_1 + x_2))x_1 = \frac{1}{2}(a - bx_1)x_1.$$
>
> This is a concave quadratic function. By the first-order condition, the opti-

mal production plan of firm 1 is

$$a - 2bx_1 = 0 \iff x_1 = \frac{a}{2b}.$$

Then the optimal production plan of firm 2 is

$$x_2 = \frac{a - bx_1}{2b} = \frac{a}{4b}.$$

7. Consider a single investor living for T periods. The utility function is

$$U_T(c_0, \cdots, c_T) = \mathrm{E} \sum_{t=0}^{T} \beta^t \log c_t,$$

where $0 < \beta < 1$ is the discount factor, c_t is consumption at time t, and "E" denotes the expectation. The initial wealth of the investor is $w > 0$. Suppose that the investor can invest money in two assets, a stock and a bond. The bond is risk-free, and the gross risk-free rate is $R_f > 0$. (For example, if the interest rate is 3%, then $R_f = 1.03$.) The gross return on stock takes two values, R_u and R_d, with probability 1/2, independent across time. Assume $R_u > R_f > R_d$ and $\frac{1}{2}(R_u + R_d) > R_f$. Let $\theta_t \geq 0$ be the fraction of wealth invested in the stock. The goal of the investor is to maximize the expected lifetime utility.

(a) (2 points) What are the state and control variables?

Solution: The state variables are wealth w and time to go T. The control variables are consumption c_t and portfolio θ_t.

(b) (2 points) If the current wealth is w, consumption is c, fraction of wealth θ is invested in stock, and the stock return is R_s ($s = u, d$), then what is the next period's wealth?

Solution: Since remaining wealth is $w - c$ and fraction θ is invested in stock (hence $1 - \theta$ is in bond), if the stock return is R_s, the next period's wealth is

$$w' = (R_s\theta + R_f(1 - \theta))(w - c).$$

(c) (3 points) Write down the Bellman equation.

Solution: Let w be the current wealth, and w' be the next period's wealth. Let $V_T(w)$ be the value function when there are T periods to go and the current wealth is w. Then the Bellman equation is

$$\begin{aligned}V_T(w) &= \max_{c,\theta} [\log c + \beta \operatorname{E}[V_{T-1}(w')]] \\ &= \max_{c,\theta} \left[\log c + \beta \sum_{s=u,d} \frac{1}{2} V_{T-1}((R_s\theta + R_f(1-\theta))(w-c))\right].\end{aligned}$$

(d) (4 points) Show that the value function takes the form $V_T(w) = a_T + b_T \log w$ for some constants a_T, b_T. (Hint: mathematical induction.)

Solution: Similar to the optimal savings problem.

(e) (4 points) Compute the optimal consumption and portfolio at $t = 0$ when $T = 1$. For notational simplicity, use $g_u := R_u - R_f > 0$ and $g_d := R_d - R_f < 0$.

Solution: Clearly $V_0(w) = \log w$. By the Bellman equation, we get

$$\begin{aligned}V_1(w) &= \max_{c,\theta} \left[\log c + \beta \sum_{s=u,d} \frac{1}{2} \log((R_s\theta + R_f(1-\theta))(w-c))\right] \\ &= \max_{c,\theta} \left[\log c + \beta \log(w-c) + \frac{\beta}{2} \sum_{s=u,d} \log(R_s\theta + R_f(1-\theta))\right].\end{aligned}$$

Note that c enters only the first two terms, and θ enters only the last term. Since $\log(\cdot)$ is concave, it is a concave maximization problem. The first-order condition with respect to c is

$$0 = \frac{1}{c} + \beta \frac{-1}{w-c} \iff c = \frac{w}{1+\beta}.$$

Ignoring $\beta/2$, the first-order condition with respect to θ is

$$\begin{aligned}&0 = \sum_{s=u,d} \frac{R_s - R_f}{R_s\theta + R_f(1-\theta)} = \sum_{s=u,d} \frac{g_s}{R_f + g_s\theta} \\ &\iff g_u(R_f + g_d\theta) + g_d(R_f + g_u\theta) = 0 \iff \theta = -\frac{R_f}{2}\frac{g_u + g_d}{g_u g_d} > 0.\end{aligned}$$

8. Suppose that there are two assets, the market portfolio and another risky asset. The gross return of the market portfolio and the risky asset are denoted by R_m, R_j. Suppose that there are three states of the world. The probability of each state and the gross returns in each state are as in the following table.

	Probability	R_m	R_j
State 1	1/4	1.2	1.3
State 2	1/2	1.1	1.2
State 3	1/4	0.8	0.7

(a) (2 points) Compute the expected returns $\mathrm{E}[R_m]$ and $\mathrm{E}[R_j]$.

Solution: It is easier to think in percent.

$$\begin{aligned} \mathrm{E}[R_m] &= \frac{1}{4}20 + \frac{1}{2}10 + \frac{1}{4}(-20) = 5\%, \\ \mathrm{E}[R_j] &= \frac{1}{4}30 + \frac{1}{2}20 + \frac{1}{4}(-30) = 10\%. \end{aligned}$$

(b) (2 points) Compute the beta of the risky asset.

Solution: The beta is defined by $\beta_j = \mathrm{Cov}[R_m, R_j]/\mathrm{Var}[R_m]$. Since beta is dimensionless, it does not matter in what unit we compute. Therefore we use percent. Then

$$\begin{aligned} \mathrm{Var}[R_m] &= \frac{1}{4}15^2 + \frac{1}{2}5^2 + \frac{1}{4}25^2 = 225, \\ \mathrm{Cov}[R_m, R_j] &= \frac{1}{4}15 \cdot 20 + \frac{1}{2}5 \cdot 10 + \frac{1}{4}(-25)(-40) = 350. \end{aligned}$$

Therefore the beta is

$$\beta_j = \frac{\mathrm{Cov}[R_m, R_j]}{\mathrm{Var}[R_m]} = \frac{350}{225} = \frac{14}{9}.$$

(c) (3 points) Compute the risk-free rate.

Solution: By the covariance pricing formula, we have

$$\mathrm{E}[R_j] - R_f = \beta_j(\mathrm{E}[R_m] - R_f).$$

Therefore the risk-free rate is

$$R_f = \frac{\beta_j \mathrm{E}[R_m] - \mathrm{E}[R_j]}{\beta_j - 1} = \frac{\frac{14}{9}5 - 10}{\frac{14}{9} - 1} = -4\%.$$

9. Consider the optimization problem

$$\begin{aligned} \text{maximize} \quad & \sum_{n=1}^{N} 2\alpha_n \sqrt{x_n} \\ \text{subject to} \quad & \sum_{n=1}^{N} p_n x_n \le w, \end{aligned}$$

where $x_1, \cdots, x_n > 0$ are variables and $w > 0$, $\alpha_1, \ldots, \alpha_N > 0$, and $p_1, \ldots, p_N > 0$ are constants.

(a) (1 point) Write down the Lagrangian.

Solution: Letting $\lambda \ge 0$ be the Lagrange multiplier on the inequality constraint, the Lagrangian is

$$L(x_1, \cdots, x_N, \lambda) = \sum_{n=1}^{N} 2\alpha_n \sqrt{x_n} + \lambda \left(w - \sum_{n=1}^{N} p_n x_n \right).$$

(b) (1 point) Fix $\alpha, p > 0$ and $\lambda \ge 0$ and let $g(x) = 2\alpha\sqrt{x} - \lambda p x$ for $x > 0$. Is $g(x)$ convex, concave, or neither?

Solution: Since

$$\begin{aligned} g'(x) &= \alpha x^{-\frac{1}{2}} - \lambda p, \\ g''(x) &= -\frac{1}{2}\alpha x^{-\frac{3}{2}} < 0, \end{aligned}$$

g is concave.

(c) (4 points) Derive the dual objective function of the original problem.

Solution: By definition, the dual objective function is

$$\omega(\lambda) = \max_{x} L(x, \lambda) = \max_{x_1, \cdots, x_N} \left[w\lambda + \sum_{n=1}^{N} (2\alpha_n \sqrt{x_n} - \lambda p_n x_n) \right].$$

Since inside the bracket is additively separable, it suffices to maximize term by term. But the summand is the same as $g(x_n)$, except that we drop the n subscript.

Now $g'(x) = 0 \iff x = (\frac{\alpha}{\lambda p})^2$, and the maximum value is $\frac{\alpha^2}{\lambda p}$, so the dual objective function is

$$\omega(\lambda) = w\lambda + \frac{1}{\lambda} \sum_{n=1}^{N} \frac{\alpha_n^2}{p_n}.$$

(d) (2 points) Derive the dual problem of the original problem.

Solution: Since the original problem is a maximization problem with an inequality constraint, the dual is a minimization problem. Therefore the dual problem is

$$\begin{array}{ll} \text{minimize} & w\lambda + \dfrac{1}{\lambda}\displaystyle\sum_{n=1}^{N} \dfrac{\alpha_n^2}{p_n} \\ \text{subject to} & \lambda \geq 0. \end{array}$$

(e) (2 points) Solve the dual problem.

Solution: Let $c = \sum_{n=1}^{N} \frac{\alpha_n^2}{p_n}$. Then the dual objective function is $\omega(\lambda) = w\lambda + \frac{c}{\lambda}$, which is convex. The first-order condition is

$$0 = \omega'(\lambda) = w - \frac{c}{\lambda^2} \iff \lambda = \sqrt{\frac{c}{w}}.$$

Therefore the solution to the dual problem is

$$\lambda = \sqrt{\frac{1}{w}\sum_{n=1}^{N} \frac{\alpha_n^2}{p_n}}.$$