Operations Research (B)
Midterm Exam 1

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January 23, 2014

Name: ______________________________

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \geq 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \geq 0$.
- The exam time is 80 minutes.
- This exam has 5 questions on 6 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	Total
Points:	20	25	10	20	25	100
Score:						

1. (a) (5 points) What is your name? Legibly write down your full name as registered to UCSD at the designated area on the cover of this exam sheet.
   (b) (5 points) Compute the inner product of the vectors $a = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ and $b = \left[\begin{smallmatrix}-3\\1\end{smallmatrix}\right]$.

Solution:

$$a \cdot b = 1 \times (-3) + 2 \times 1 = -3 + 2 = -1.$$

You can also use the notation $\langle a, b \rangle$ for the inner product.

   (c) (5 points) Define the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and the vector $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Compute Ab.

Solution:

$$Ab = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times (-1) \\ (-1) \times 1 + 3 \times (-1) \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}.$$

   (d) (5 points) Consider the following numerical algorithms: the bisection method, the Newton method, and the false position method. Which algorithm is the fastest? Which is the slowest?

Solution: The order of convergence is 1 for bisection, 2 for Newton, and 1.618 for false position, so the fastest is Newton and the slowest is bisection.

2. For each of the following functions, compute the first and second derivatives, and determine whether it is convex, concave, or neither. (2 point each for derivatives, 1 point for convexity.)
   (a) (5 points) $f(x) = 9x - 2x^2$.

Solution: $f'(x) = 9 - 4x$, $f''(x) = -4 < 0$, so f is concave.

   (b) (5 points) $f(x) = x^5 - 17x$.

Solution: $f'(x) = 5x^4 - 17$, $f''(x) = 20x^3$, which can be positive or negative. Therefore f is neither convex nor concave.

   (c) (5 points) $f(x) = x^6 - 15x^4 + 135x^2 - 31x + 2015$.

Solution:

$$\begin{aligned} f'(x) &= 6x^5 - 60x^3 + 270x - 31, \\ f''(x) &= 30x^4 - 180x^2 + 270 = 30(x^4 - 6x^2 + 9) = 30(x^2 - 3)^2 \geq 0, \end{aligned}$$

so f is convex.

(d) (5 points) $f(x) = \sqrt{x^2+1}$.

> **Solution:**
>
> $$f'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}},$$
> $$f''(x) = \frac{1 \cdot \sqrt{x^2+1} - x \cdot \frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{1}{(x^2+1)^{\frac{3}{2}}} > 0,$$
>
> so f is convex.

(e) (5 points) $f(x) = \log(\log x)$, where $x > 1$.

> **Solution:** By the chain rule, we get
>
> $$f'(x) = \frac{1}{\log x}\frac{1}{x} = \frac{1}{x \log x},$$
> $$f''(x) = -\frac{(x\log x)'}{(x\log x)^2} = -\frac{\log x + 1}{(x \log x)^2}.$$
>
> Since $x > 1$, we have $\log x > 0$, so $f''(x) < 0$. Therefore f is concave.

3. A firm produces an output. The unit price of the output is $p > 0$. If a firm chooses output level x, it costs $\frac{1}{2}cx^2$ for production, where $c > 0$.

(a) (5 points) Suppose you are the manager of the firm and you want to maximize the profit. Write down the objective function of this problem.

> **Solution:** If you produce x, the profit is
>
> $$f(x) = px - \frac{1}{2}cx^2,$$
>
> which is the objective function.

(b) (5 points) Determine the optimal production level.

> **Solution:** Since $f'(x) = p - cx$ and $f''(x) = -c < 0$, f is concave. Therefore f attains the maximum at
>
> $$f'(x) = 0 \iff p - cx = 0 \iff x = \frac{p}{c}.$$

4. Let $f(x_1, x_2) = x_1^2 - x_1x_2 + 2x_2^2 - x_1 - 3x_2$.

(a) (4 points) Compute the gradient of f.

Solution:

$$\nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 - 1 \\ -x_1 + 4x_2 - 3 \end{bmatrix}.$$

(b) (4 points) Compute the Hessian of f.

Solution:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}.$$

(c) (4 points) Determine whether f is convex, concave, or neither.

Solution: Let Q be the Hessian of f (the above matrix). Let $h = \left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right]$ be any nonzero vector. Then

$$\begin{aligned} \langle h, Qh \rangle &= \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &= \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} 2h_1 - h_2 \\ -h_1 + 4h_2 \end{bmatrix} \\ &= 2h_1^2 - 2h_1 h_2 + 4h_2^2 \\ &= 2\left(h_1 - \frac{1}{2}h_2\right)^2 + \frac{7}{2}h_2^2 > 0, \end{aligned}$$

so Q is positive definite. Therefore f is convex.

(d) (4 points) Find the stationary point(s) of f.

Solution:

$$\nabla f(x) = 0 \iff \begin{bmatrix} 2x_1 - x_2 - 1 \\ -x_1 + 4x_2 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so the stationary point of f is $(x_1, x_2) = (1, 1)$.

(e) (4 points) Determine whether each stationary point is a maximum, minimum, or neither.

Solution: Since f is convex, any stationary point is a minimum. Therefore the point $(x_1, x_2) = (1, 1)$ is the minimum of f.

5. Let $f(x) = \log(\mathrm{e}^x + \mathrm{e}^{-x})$.

(a) (4 points) Compute $f'(x)$ and $f''(x)$.

Solution:

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$
$$f''(x) = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}.$$

(b) (4 points) Is f convex, concave, or neither?

Solution: Since $f''(x) > 0$, it is convex.

(c) (4 points) What is the minimum of f?

Solution:

$$0 = f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \iff e^x = e^{-x} \iff x = 0,$$

so the minimum of f is $x^* = 0$.

(d) (5 points) Suppose you want to minimize f by using the Newton algorithm. Let x_n be the approximate solution at iteration n. Express x_{n+1} using x_n.

Solution:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} = x_n - \frac{1}{4}(e^{2x_n} - e^{-2x_n}).$$

(e) (8 points) Suppose $x_0 > 3$. Show that $|x_{n+1}| > |x_n|$ and that the Newton algorithm does not converge to the minimum of f. (Hint: $e^{-a} < 1 < 1 + a + \frac{a^2}{2} < e^a$ if $a > 0$.)

Solution: Assume $x_n = x > 0$ (the argument is the same if $x < 0$). Using the hint for $a = 2x$, we get the inequality

$$\begin{aligned} x_{n+1} &= x - \frac{1}{4}(e^{2x} - e^{-2x}) = x - \frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} \\ &< x - \frac{1}{4}\left(1 + (2x) + \frac{1}{2}(2x)^2\right) + \frac{1}{4} \\ &= \frac{1}{2}x - \frac{1}{2}x^2. \end{aligned}$$

If $x_n = x > 3$, then

$$x_{n+1} < \frac{1}{2}x - \frac{1}{2}x^2 < \frac{1}{2}x - \frac{1}{2}3x = -x = -|x_n|.$$

Therefore $|x_{n+1}| > |x_n|$. Since the absolute value of x_n increases, it does not converge to the solution $x^* = 0$.

You can detach this sheet and use it as a scratch paper.