Operations Research (B)
Midterm Exam 2

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Name: ______________________________

Instruction:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" is synonymous to "prove".
- Full credit will not be given to correct but unsupported claims. Example: $x^2 - 2x + 1 \ge 0$ is true but not obvious. You need to argue $x^2 - 2x + 1 = (x-1)^2 \ge 0$.
- The exam time is 80 minutes.
- This exam has 5 questions on 7 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	Total
Points:	20	20	20	15	25	100
Score:						

1. Suppose that there are two goods and a consumer with utility function

$$u(x_1, x_2) = \log x_1 + 2 \log x_2,$$

where $x_1, x_2 > 0$. Suppose that the consumer has wealth $w > 0$ and the price of each good is $p_1, p_2 > 0$.

(a) (5 points) If the consumer consumes x_1 of good 1 and x_2 of good 2, what is the expenditure?

Solution: $p_1 x_1 + p_2 x_2$.

(b) (5 points) Write down the Lagrangian for the utility maximization problem.

Solution: The budget constraint is $p_1 x_1 + p_2 x_2 \leq w$. Since it is a maximization problem, the inequality should be $w - p_1 x_1 - p_2 x_2 \geq 0$. Therefore the Lagrangian is

$$L(x_1, x_2, \lambda) = \log x_1 + 2 \log x_2 + \lambda(w - p_1 x_1 - p_2 x_2).$$

(c) (4 points) Explain why the Karush-Kuhn-Tucker conditions are both necessary and sufficient for optimality.

Solution: It is necessary because with linear constraints, constraint qualifications are automatically satisfied (2 points). It is sufficient because it is a concave maximization problem (2 points).

(d) (6 points) Solve the utility maximization problem.

Solution: The first-order conditions are

$$0 = \frac{\partial L}{\partial x_1} = \frac{1}{x_1} - \lambda p_1,$$
$$0 = \frac{\partial L}{\partial x_2} = \frac{2}{x_2} - \lambda p_2.$$

Solving these we get $x_1 = \frac{1}{\lambda p_1}$, $x_2 = \frac{2}{\lambda p_2}$. By the complementary slackness, we have

$$0 = \lambda(w - p_1 x_1 - p_2 x_2) = \lambda(w - 3/\lambda) \iff \lambda = \frac{3}{w}.$$

Therefore the solution is $(x_1, x_2) = \left(\frac{w}{3p_1}, \frac{2w}{3p_2}\right)$.

2. Consider the problem

$$\begin{aligned} &\text{minimize} && 3x_1 + x_2^3 \\ &\text{subject to} && x_1^2 + x_2^2 \leq 0. \end{aligned}$$

(a) (5 points) Compute the solution.

Solution: Since the only feasible point is $(x_1, x_2) = (0, 0)$, it is the solution.

(b) (5 points) Write down the Lagrangian.

Solution:

$$L(x_1, x_2, \lambda) = 3x_1 + x_2^3 + \lambda(x_1^2 + x_2^2).$$

(c) (5 points) Does the Karush-Kuhn-Tucker (KKT) conditions hold? Answer yes or no. If yes, give the Lagrange multiplier. If no, prove it.

Solution: No (2 points). Suppose that the KKT conditions hold at the solution $(x_1, x_2) = (0, 0)$. Then by the first order condition with respect to x_1, we get

$$0 = \frac{\partial L}{\partial x_1} = 3 + 2\lambda x_1 = 3,$$

which is a contradiction (3 points).

(d) (5 points) Explain why the KKT conditions do or do not hold in the previous question.

Solution: Let $g(x_1, x_2) = x_1^2 + x_2^2$, which is convex. Then the condition $x_1^2 + x_2^2 \leq 0$ is equivalent to $g(x_1, x_2) \leq 0$. Since $g(x_1, x_2) \geq 0$ always, there are no points such that $g(x_1, x_2) < 0$. Since the Slater constraint qualification does not hold, KKT conditions need not hold. (3 points for mentioning constraint qualification, 2 points for showing that SCQ or LICQ fail.)

3. Consider the problem

$$\begin{array}{ll} \text{maximize} & x_1^2 + x_1 x_2 + x_2^2 \\ \text{subject to} & x_1^2 + x_2^2 \leq 1. \end{array}$$

(a) (3 points) Are the Karush-Kuhn-Tucker conditions necessary for a solution? That is, does the solution to the problem have to satisfy the KKT conditions? Answer yes or no, then explain why.

Solution: Yes (1 points). Since the constraints are convex, and the Slater constraint qualification holds for $(x_1, x_2) = (0, 0)$, the KKT theorem applies (2 points).

(b) (3 points) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? That is, does a point satisfying the KKT conditions have to be a solution to the problem? Answer yes or no, then explain why.

Solution: No (1 points). Since the objective function is convex but it is a maximization problem, a point satisfying the KKT conditions need not be a solution (2 points).

(c) (4 points) Write down the Lagrangian.

Solution:

$$L(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2 + \lambda(1 - x_1^2 - x_2^2).$$

(d) (5 points) Derive the first-order condition with respect to x_1 and express x_2 using x_1 and the Lagrange multiplier.

Solution: Since the objective function is continuous and the constraint set is closed and bounded, a solution exist. The first-order conditions are

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} = 2x_1 + x_2 - 2\lambda x_1, \\ 0 &= \frac{\partial L}{\partial x_2} = x_1 + 2x_2 - 2\lambda x_2. \end{aligned}$$

From the first equation we get $x_2 = (2\lambda - 2)x_1$. (3 points for FOC, 2 points for the rest.)

(e) (5 points) Compute the solution.

Solution: Substituting $x_2 = (2\lambda - 2)x_1$ into the second equation, we get

$$0 = x_1(1 - (2\lambda - 2)^2) \iff x_1 = 0, \lambda = \frac{1}{2}, \frac{3}{2}.$$

If $x_1 = 0$, then $x_2 = (2\lambda - 2)x_1 = 0$, so the solution is $(x_1, x_2) = (0, 0)$. If $\lambda = \frac{1}{2}$, then $x_2 = (2\lambda - 2)x_1 = -x_1$. By complementary slackness we have

$$0 = 1 - x_1^2 - x_2^2 \iff x_1^2 = \frac{1}{2} \iff x_1 = \pm\frac{1}{\sqrt{2}},$$

so the solution is $(x_1, x_2) = (1/\sqrt{2}, -1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2})$. If $\lambda = \frac{3}{2}$, then $x_2 = (2\lambda - 2)x_1 = x_1$. By complementary slackness, we have

$$0 = 1 - x_1^2 - x_2^2 \iff x_1^2 = \frac{1}{2} \iff x_1 = \pm\frac{1}{\sqrt{2}},$$

so the solution is $(x_1, x_2) = (1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, -1/\sqrt{2})$. Letting

$$f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$$

be the objective function, we have

$$\begin{aligned} f(0,0) &= 0, \\ f(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}) &= \frac{3}{2}, \\ f(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}) &= \frac{1}{2}, \end{aligned}$$

so the maximum value is $\frac{3}{2}$ and the solution is

$$(x_1, x_2) = \left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right).$$

4. There is a stair consisting of N steps. At each time, you can proceed i steps, incurring cost (effort) c_i, where $i = 1, 2, \dots, I$. You want to climb the stair in minimum effort.
   (a) (5 points) What is the state variable? What is the control variable?

   Solution: The state variable is N (3 points). The control variable is i (2 points).

   (b) (5 points) Write down the Bellman equation.

   Solution: If you proceed i steps, you incur cost c_i and there are $N - i$ steps to go. Therefore the Bellman equation is

   $$V(N) = \min_i \{c_i + V(N-i)\}.$$

   (c) (5 points) Solve the problem for $N = 8$ when $I = 3$ and $(c_1, c_2, c_3) = (2, 3, 4)$.

   Solution: Clearly $V(0) = 0$ and $V(1) = 2$. Iterating the Bellman equation, we get $V(2) = 3$, $V(3) = 4$, $V(4) = 6$, $V(5) = 7$, $V(6) = 8$, $V(7) = 10$, $V(8) = 11$.

5. Consider a row of N coins of values $v_1, \dots, v_N$. You play a game against an opponent by alternating turns. At each turn, a player selects either the first or last coin from the row, removes it from the row permanently, and receives the value of the coin.

You want to determine the optimal strategy to play this game. Let $V(v_1, \dots, v_N)$ be your maximum value when you move first.

(a) (5 points) If you pick coin 1 first, what is the maximum possible amount of money your opponent can get?

Solution: If you pick coin 1 first, the opponent is faced with the same problem with coins of value $v_2, \dots, v_N$, so your opponent will get $V(v_2, \dots, v_N)$.

(b) (5 points) If you pick coin 1 first, what is the maximum possible amount of money you can get?

Solution: Since all coins are held by someone, your value will be

$$v_1 + \cdots + v_N - V(v_2, \dots, v_N).$$

(c) (8 points) Write down the Bellman equation.

Solution: If you pick coin N first, by the same argument your value will be

$$v_1 + \cdots + v_N - V(v_1, \dots, v_{N-1}).$$

Therefore the Bellman equation is

$$V(v_1, \dots, v_N) = v_1 + \cdots + v_N - \min \{V(v_1, \dots, v_{N-1}), V(v_2, \dots, v_N)\}.$$

(d) (7 points) Describe the optimal strategy of you and your opponent when $N = 4$ and $(v_1, v_2, v_3, v_4) = (5, 7, 3, 2)$, assuming that you move first.

Solution: If there is one coin, there is no choice but to take the coin, so $V(v_1) = v_1$. By the Bellman equation, we have

$$V(v_1, v_2) = v_1 + v_2 - \min \{v_1, v_2\} = \max \{v_1, v_2\}.$$

Therefore

$$\begin{aligned} V(5, 7, 3) &= 5 + 7 + 3 - \min \{V(5, 7), V(7, 3)\} = 15 - \min \{7, 7\} = 8, \\ V(7, 3, 2) &= 5 + 7 + 3 - \min \{V(7, 3), V(3, 2)\} = 15 - \min \{7, 3\} = 12, \\ V(5, 7, 3, 2) &= 5 + 7 + 3 + 2 - \min \{V(5, 7, 3), V(7, 3, 2)\} = 17 - \min \{8, 12\} = 9. \end{aligned}$$

The maximum value you can get when you move first is therefore 9. The optimal strategy is to take the last coin (value 2) first. Then your opponent will take the coin with value 5, you take 7, and opponent 3.