Econ 200B General Equilibrium
Midterm Exam

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Name: ______________________________

Instructions:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" or "prove" require a formal mathematical proof or argument. "Explain" only requires an intuitive, though logically coherent, verbal argument.
- This exam has 5 questions on 10 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	Total
Points:	15	15	15	15	40	100
Score:						

1. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility function of each agent is

$$u_A(x_1, x_2) = x_1 - \frac{8}{x_2},$$
$$u_B(x_1, x_2) = x_1 + 10 \log x_2.$$

Suppose that the initial endowments are $(e_1^A, e_2^A) = (4, 4)$ and $(e_1^B, e_2^B) = (10, 3)$.

(a) (2 points) What is the name of these utility functions?

Solution: Quasi-linear.

(b) (8 points) Consider the problem of maximizing the sum of utilities,

$$\begin{aligned} &\text{maximize} && u_A(x_1^A, x_2^A) + u_B(x_1^B, x_2^B) \\ &\text{subject to} && x_1^A + x_1^B \le e_1^A + e_1^B, \\ &&& x_2^A + x_2^B \le e_2^A + e_2^B. \end{aligned}$$

Let $(x_1^A, x_2^A, x_1^B, x_2^B)$ be a solution. Compute x_2^A and x_2^B.

Solution: Since the utility functions are quasi-linear, it suffices to maximize the sum of the nonlinear part. For notational simplicity, let $x_2^A = y$ and $x_2^B = z$. Then the problem becomes

$$\begin{aligned} &\text{maximize} && -\frac{8}{y} + 10 \log z \\ &\text{subject to} && y + z \le 4 + 3 = 7. \end{aligned}$$

The Lagrangian is

$$L(y, z, \lambda) = -\frac{8}{y} + 10 \log z + \lambda(7 - y - z).$$

The first-order conditions are

$$0 = \frac{\partial L}{\partial y} = \frac{8}{y^2} - \lambda = 0 \iff y = \sqrt{\frac{8}{\lambda}},$$
$$0 = \frac{\partial L}{\partial z} = \frac{10}{z} - \lambda = 0 \iff z = \frac{10}{\lambda}.$$

Substituting into the feasibility constraint, we get

$$y + z = 7 \iff \sqrt{\frac{8}{\lambda}} + \frac{10}{\lambda} = 7.$$

Letting $t = \sqrt{\frac{2}{\lambda}}$, this equation becomes

$$2t + 5t^2 = 7 \iff 5t^2 + 2t - 7 = 0 \iff (5t + 7)(t - 1) = 0 \iff t = 1$$

since $t > 0$. Hence $\lambda = \frac{2}{t^2} = 2$. Therefore $x_2^A = y = \sqrt{8/\lambda} = 2$ and $x_2^B = z = 10/\lambda = 5$.

(c) (5 points) Let the price be $p_1 = 1$ and $p_2 = p$. Compute the equilibrium price p and the allocation.

Solution: In a quasi-linear economy, we know from the lecture notes that the price is equal to the Lagrange multiplier. Therefore $p = \lambda = 2$, and the consumption of good 2 is the same as above. By the budget constraint, the consumption of good 1 is

$$\begin{aligned} x_1^A &= e_1^A + p(e_2^A - x_2^A) = 4 + 2(4 - 2) = 8, \\ x_1^B &= e_1^B + p(e_2^B - x_2^B) = 10 + 2(3 - 5) = 6. \end{aligned}$$

2. Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy with I agents and L goods.

(a) (5 points) What is the first welfare theorem? State the assumptions as well as the conclusion.

Solution: The first welfare theorem says that if utility functions are locally nonsatiated, then the competitive equilibrium allocation is Pareto efficient.

(b) (10 points) Prove the first welfare theorem.

Solution: See the lecture note.

3. Consider an economy with I agents and L basic goods labeled by $l = 1, \dots, L$. Suppose that there is another good, labeled 0, which is a public good. (A public good is non-excludable, *i.e.*, the consumption of one agent does not reduce the availability of that good to other agents. Therefore all agents consume the same amount of good 0, which equals aggregate supply in equilibrium.) Suppose that there are no endowments of good 0, which is produced from other goods $l = 1, \dots, L$ using some technology represented by a production function $y = f(x_1, \dots, x_L)$. It is well known that the presence of a public good may make the economy inefficient. Let $u_i(x_0, x_1, \dots, x_L)$ be the utility function of agent i, assumed to be locally nonsatiated.

(a) (7 points) Show that by quoting an individual-specific price for the public good, we can make the competitive equilibrium allocation efficient. (Hint: expand the set of goods, and consider an economy with $L + I$ goods labeled by $l = 1, \dots, L + I$. Goods $l = 1, \dots, L$ are the L basic goods, and good $L + i$ is the public good consumed by agent i. Make sure to discuss how we should reinterpret the production technology and market clearing conditions.)

Solution: Let $p = (p_1, \dots, p_{L+I})$ be the price vector. Define the utility function of agent i by

$$U_i(x_1, \dots, x_{L+I}) = u_i(x_{L+i}, x_1, \dots, x_L).$$

Let $y = (y_1, \dots, y_{L+I})$ be an input-output vector. Since y_{L+i} is the output of the public good consumed by agent i, it must be the case that

$$y_{L+1} = \cdots = y_{L+I} \leq f(-y_1, \dots, -y_L).$$

The market clearing condition for good $l \leq L$ is

$$\sum_{i=1}^{I} x_{il} \leq \sum_{i=1}^{I} e_{il} + y_l,$$

as usual. The market clearing condition for good $L + i$ is

$$x_{i,L+i} = y_{L+i}.$$

Under this interpretation, the economy becomes a standard Arrow-Debreu economy, so the equilibrium allocation is Pareto efficient.

(b) (8 points) Suppose that there are two agents ($i = 1, 2$), one basic good, and a public good. Agent i has utility function

$$u_i(x_0, x_1) = \alpha_i \log x_0 + (1 - \alpha_i) \log x_1,$$

where x_0, x_1 are consumption of the public good and the basic good. Let e_i be the initial endowment of agent i's basic good, and suppose there is a technology that converts the basic good to the public good one-for-one. Normalize the price of the basic good to be 1. Find individual-specific prices for the public good to make the competitive equilibrium allocation efficient.

Solution: Let p_i be the price of public good charged to agent i. If the firm produces y units of the public good, since technology is one-for-one, the input is also y. Therefore the revenue is $p_1 y + p_2 y = (p_1 + p_2)y$, and the profit is $(p_1 + p_2)y - y = (p_1 + p_2 - 1)y = 0$. By profit maximization, it must be $p_1 + p_2 = 1$. Since utilities are Cobb-Douglas, the demand of agent i is

$$(x_{i0}, x_{i1}) = \left(\frac{\alpha_i e_i}{p_i}, (1 - \alpha_i) e_i \right).$$

Hence the market clearing conditions for goods 1, 2, 3 are

$$\begin{aligned} (1 - \alpha_1)e_1 + (1 - \alpha_2)e_2 &= e_1 + e_2 - y, \\ \frac{\alpha_1 e_1}{p_1} &= y, \\ \frac{\alpha_2 e_2}{p_2} &= y. \end{aligned}$$

Solving these equations we obtain $y = \alpha_1 e_1 + \alpha_2 e_2$ and $p_i = \frac{\alpha_i e_i}{\alpha_1 e_1 + \alpha_2 e_2}$.

4. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. All agents have utility function

$$u(x_1, x_2) = x_1 x_2.$$

Suppose that the initial endowments are $e_1 = (16, 1)$, $e_2 = (4, 4)$, and $e_3 = (10, 25)$. Below, always normalize the price of good 1 to be 1.

(a) (5 points) Compute the equilibrium price, allocation, and utility levels when country A is in autarky.

Solution: Since agents have identical homothetic utility, it is a representative agent economy. Since the aggregate endowment in country A is $e_1 + e_2 = (20, 5)$, the prices are proportional to the marginal utilities, which are $(x_2, x_1) = (5, 20)$. Letting p be the price of good 2, we have $p = 20/5 = 4$. The wealth share of agent 1 is

$$\frac{1 \cdot 16 + 4 \cdot 1}{1 \cdot 20 + 4 \cdot 5} = \frac{1}{2}.$$

Therefore the equilibrium allocation is

$$(x_{11}, x_{12}) = (x_{21}, x_{22}) = \frac{1}{2}(20, 5) = \left(10, \frac{5}{2}\right).$$

The utility levels are $10 \times \frac{5}{2} = 25$ for both agents.

(b) (5 points) Compute the free trade equilibrium price, allocation, and utility levels of agents 1 and 2.

Solution: Since the world aggregate endowment is $e = e_1 + e_2 + e_3 = (30, 30)$, by the same argument as above we have $p = 1$. The wealth share of each agent is

$$w_1 = \frac{1 \cdot 16 + 1 \cdot 1}{1 \cdot 30 + 1 \cdot 30} = \frac{17}{60},$$
$$w_2 = \frac{1 \cdot 4 + 1 \cdot 4}{1 \cdot 30 + 1 \cdot 30} = \frac{8}{60},$$
$$w_3 = \frac{1 \cdot 10 + 1 \cdot 25}{1 \cdot 30 + 1 \cdot 30} = \frac{35}{60}.$$

Therefore the equilibrium allocation is

$$(x_{11}, x_{12}) = w_1 e = \left(\frac{17}{2}, \frac{17}{2}\right),$$
$$(x_{21}, x_{22}) = w_2 e = (4, 4),$$
$$(x_{31}, x_{32}) = w_3 e = \left(\frac{35}{2}, \frac{35}{2}\right).$$

The utility levels are $u_1 = \frac{289}{4}$ and $u_2 = 16$.

(c) (5 points) Concerned with agent 2's welfare, suppose that the government of country A introduces a direct tax/subsidy to its citizens. Find a tax scheme in country A that makes the free trade allocation weakly Pareto dominant to the autarky allocation.

Solution: Since the economy is a representative-agent economy, redistribution of wealth does not affect the equilibrium price. Therefore the free trade equilibrium price is still $p = 1$. To make the autarky allocation just affordable, country A should impose tax $t_1 = 16 + 1 - 10 - \frac{5}{2} = \frac{9}{2}$ on agent 1 and $t_2 = 4 + 4 - 10 - \frac{5}{2} = -\frac{9}{2}$ on agent 2. Since the after tax wealth is $10 + \frac{5}{2} = \frac{25}{2}$ for both agents, the new equilibrium allocation is

$$(x_{11}, x_{12}) = (x_{21}, x_{22}) = \left(\frac{25}{4}, \frac{25}{4}\right)$$

and (x_{31}, x_{32}) is unchanged. The utility level of both agents 1 and 2 is $(25/4)^2 > 5^2 = 25$, so both agents 1 and 2 are better off than autarky.

5. Consider an economy with two agents ($i = 1, 2$), two periods ($t = 0, 1$), and two states ($s = u, d$) at $t = 1$. The aggregate endowment is $e = (e_0, e_u, e_d)$, where $e_0 > 0$ and $e_u > e_d > 0$ (so u, d are the "up" and "down" states). The initial endowments are $e_1 = \alpha e$ and $e_2 = (1 - \alpha)e$, so the wealth share of agent 1 is $0 < \alpha < 1$. Agent 1's subjective probability of state s is $\pi_s > 0$, where $\pi_u + \pi_d = 1$. The utility functions are

$$u_1(x_0, x_u, x_d) = (1 - \beta) \log x_0 + \beta(\pi_u \log x_u + \pi_d \log x_d),$$
$$u_2(x_0, x_u, x_d) = (1 - \beta) \log x_0 + \beta \log(\min \{x_u, x_d\}),$$

where $0 < \beta < 1$ is the discount factor. Let $p = (1, p_u, p_d)$ be the price vector and $W = e_0 + p_u e_u + p_d e_d$ be the value of aggregate endowment. Note that this is an Arrow-Debreu economy, so all trades occur at $t = 0$.

(a) (5 points) Explain why there exists a unique equilibrium.

Solution: An equilibrium exists because endowments are positive and preferences are continuous, concave, and locally non-satiated. Since endowments are collinear, $\exp(u_1), \exp(u_2)$ are homogeneous of degree 1, and u_1, u_2 are concave, according to the lecture note the equilibrium is unique.

(b) (3 points) Compute the demand of agent 1.

Solution: Since agent 1 has Cobb-Douglas utility function, the demand is

$$(x_{10}, x_{1u}, x_{1d}) = \left((1-\beta)\alpha W, \frac{\alpha\beta\pi_u W}{p_u}, \frac{\alpha\beta\pi_d W}{p_d}\right).$$

(c) (3 points) Compute the demand of agent 2.

Solution: Since agent 2 has a Leontief preference over consumption at time 1, it must be $x_u = x_d = x_1$. The cost for buying this bundle is $(p_u + p_d)x_1$. Hence agent 2 maximizes

$$(1-\beta)\log x_0 + \beta \log x_1$$

subject to $x_0 + (p_u + p_d)x_1 \leq (1-\alpha)W$, which is essentially a utility maximization problem with a Cobb-Douglas utility function. Therefore the demand is

$$(x_{20}, x_{2u}, x_{2d}) = \left((1-\beta)(1-\alpha)W, \frac{(1-\alpha)\beta W}{p_u + p_d}, \frac{(1-\alpha)\beta W}{p_u + p_d}\right).$$

(d) (4 points) Let a "stock" be an asset that pays out the aggregate endowment at $t = 1$. Let a "risk-free asset" be an asset that pays out 1 at $t = 1$ no matter what. Let q_s, q_f be the price of the stock and risk-free asset at $t = 0$. Express q_s, q_f using only p_u, p_d, e_u, e_d.

Solution: Since the stock pays the aggregate endowment, by no arbitrage we have $q_s = p_u e_u + p_d e_d$. Similarly, $q_f = p_u + p_d$.

(e) (4 points) Express p_u, p_d using only q_s, q_f, e_u, e_d.

Solution: Writing the above equations in matrix form, we have

$$\begin{bmatrix} e_u & e_d \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_u \\ p_d \end{bmatrix} = \begin{bmatrix} q_s \\ q_f \end{bmatrix}$$

$$\iff \begin{bmatrix} p_u \\ p_d \end{bmatrix} = \frac{1}{e_u - e_d} \begin{bmatrix} 1 & -e_d \\ -1 & e_u \end{bmatrix} \begin{bmatrix} q_s \\ q_f \end{bmatrix} = \frac{1}{e_u - e_d} \begin{bmatrix} q_s - q_f e_d \\ -q_s + q_f e_u \end{bmatrix}.$$

(f) (5 points) Using the market clearing condition for $t = 0$, compute q_s using only exogenous parameters.

Solution: By the market clearing condition at $t = 0$, we obtain

$$e_0 = (1-\beta)\alpha W + (1-\beta)(1-\alpha)W = (1-\beta)W.$$

Since $W = e_0 + p_u e_u + p_d e_d = e_0 + q_s$, it follows that $q_s = \frac{\beta}{1-\beta} e_0$.

(g) (6 points) Using the market clearing condition for $s = u, d$, show that

$$\frac{\alpha\beta}{1-\beta} e_0 \left(\frac{\pi_u}{p_u} - \frac{\pi_d}{p_d} \right) = e_u - e_d.$$

Solution: By the market clearing condition for state s, we obtain

$$e_s = x_{1s} + x_{2s} = \frac{\alpha\beta\pi_s W}{p_s} + \frac{\beta(1-\alpha)W}{p_u + p_d}.$$

Taking the difference for $s = u, d$, we obtain

$$\alpha\beta W \left(\frac{\pi_u}{p_u} - \frac{\pi_d}{p_d} \right) = e_u - e_d.$$

Substituting $(1-\beta)W = e_0$, we obtain the desired equation.

(h) (5 points) Show that the risk-free rate goes up when agent 1 becomes relatively richer (*i.e.*, α gets larger).

Solution: Eliminating p_u, p_d from the above equation using q_s, q_f, we obtain

$$\alpha q_s \left(\frac{\pi_u}{q_s - q_f e_d} - \frac{\pi_d}{q_f e_u - q_s} \right) - 1 = 0.$$

Noting that $q_s = \frac{\beta}{1-\beta} e_0$ is constant, let $F(\alpha, q_f)$ be the left-hand side. Since in equilibrium we have $p_s > 0$, it must be $q_s - q_f e_d > 0$ and $q_f e_u - q_s > 0$. Under these conditions, we have

$$F_\alpha = q_s \left(\frac{\pi_u}{q_s - q_f e_d} - \frac{\pi_d}{q_f e_u - q_s} \right) = \frac{1}{\alpha} > 0,$$
$$F_{q_f} = \alpha q_s \left(\frac{\pi_u e_d}{(q_s - q_f e_d)^2} + \frac{\pi_d e_u}{(q_f e_u - q_s)^2} \right) > 0.$$

Hence by the implicit function theorem, $\partial q_f / \partial \alpha = -F_\alpha / F_{q_f} < 0$. Since $R_f = 1/q_f$, it follows that $\partial R_f / \partial \alpha > 0$, so when agent 1 gets relatively richer, the interest rate goes up.

(i) (5 points) Show that the risk-free rate goes up when agent 1 becomes more optimistic (*i.e.*, π_u gets larger).

Solution: Let

$$F(\pi, q_f) = \alpha q_s \left(\frac{\pi}{q_s - q_f e_d} - \frac{1-\pi}{q_f e_u - q_s} \right) - 1,$$

where $\pi = \pi_u$. Then

$$F_\pi = \alpha q_s \left(\frac{1}{q_s - q_f e_d} + \frac{1}{q_f e_u - q_s} \right) > 0,$$

so by the implicit function theorem, $\partial q_f / \partial \pi = -F_\pi / F_{q_f} < 0$.

You can detach this sheet and use as a scratch paper.