Econ 200B General Equilibrium
Midterm Exam

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Name: ____

Instructions:

- Read these instructions and the questions carefully.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, *i.e.*, base $\mathrm{e} = 2.718281828\dots$.
- "Show" or "prove" require a formal mathematical proof or argument. "Explain" only requires an intuitive, though logically coherent, verbal argument.
- This exam has 5 questions on 11 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

1. Consider an economy consisting of two agents, $i = A, B$, and two goods, $l = 1, 2$. The initial endowments are $e_A = a = (a_1, a_2) \gg 0$, $e_B = b = (b_1, b_2) \gg 0$, and agents have Cobb-Douglas utility functions

$$U_A(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2,$$
$$U_B(x_1, x_2) = \beta \log x_1 + (1 - \beta) \log x_2,$$

where $\alpha, \beta \in (0, 1)$. Below, normalize the price of good 2 to be $p_2 = 1$ and set $p_1 = p$.

(a) (3 points) Let $w = pa_1 + a_2$ be the wealth of agent A. Write down the Lagrangian of the utility maximization problem.

Solution:

$$L = \alpha \log x_1 + (1 - \alpha) \log x_2 + \lambda(w - px_1 - x_2).$$

(b) (4 points) Derive the first-order conditions.

Solution:

$$\frac{\alpha}{x_1} - \lambda p = 0,$$
$$\frac{1 - \alpha}{x_2} - \lambda = 0.$$

(c) (4 points) Express the demand of agent A using only α, a_1, a_2, p.

Solution: Since the objective function is concave, the first-order condition and complementary slackness condition are sufficient for optimality. After some algebra we obtain

$$(x_1, x_2) = \left(\frac{\alpha(pa_1 + a_2)}{p}, (1 - \alpha)(pa_1 + a_2) \right).$$

(d) (4 points) Compute the equilibrium price.

Solution: Using the market clearing condition for good 2, we obtain

$$(1 - \alpha)(pa_1 + a_2) + (1 - \beta)(pb_1 + b_2) = a_2 + b_2$$
$$\iff p = \frac{\alpha a_2 + \beta b_2}{(1 - \alpha)a_1 + (1 - \beta)b_1}.$$

(e) (5 points) Show that if either (i) the supply of good 1 decreases (a_1 or b_1 goes down), (ii) the supply of good 2 increases (a_2 or b_2 goes up), or (iii) agents like good 1 more (α or β goes up), then the price of good 1 increases.

Solution: Since $a_1, a_2, b_1, b_2 > 0$ and $\alpha, \beta \in (0, 1)$, the above expression for p is increasing in a_2, b_2, α, β and decreasing in a_1, b_1.

2. Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. Suppose that for all i we have $e_i \gg 0$ and the utility function $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ is continuous, quasi-concave, and locally nonsatiated. Suppose in addition that for some i, u_i is weakly monotonic.
   (a) (3 points) What is the definition of local nonsatiation?

   Solution: u_i is locally nonsatiated if for any $x \in \mathbb{R}^L_+$ and $\epsilon > 0$, there exists $y \in \mathbb{R}^L_+$ such that $\|y - x\| < \epsilon$ and $u_i(y) > u_i(x)$.

   (b) (3 points) What is the definition of weak monotonicity?

   Solution: u_i is weakly monotonic if for any $x, y \in \mathbb{R}^L_+$, $x \leq y$ implies $u(x) \leq u(y)$ and $x \ll y$ implies $u(x) < u(y)$.

   (c) (14 points) Show that $\mathcal{E}$ has a competitive equilibrium $\{p, (x_i)\}$ in which all markets clear exactly, that is,

$$\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i.$$

   You can use (without proofs) the existence results explained in the lecture note.

   Solution: According to the lecture note, under the maintained assumptions there exists a competitive equilibrium $\{p, (x_i)\}$. By market clearing, we have

$$\sum_{i=1}^{I} (x_i - e_i) \leq 0.$$

   Without loss of generality, assume that agent 1 is weakly monotonic. Consider the allocation (x'_i), where $x'_i = x_i$ for all $i \geq 2$ and

$$x'_1 = \sum_{i=1}^{I} e_i - \sum_{i=2}^{I} x'_i = \sum_{i=1}^{I} e_i - \sum_{i=2}^{I} x_i \geq x_1.$$

   Let us show that $\{p, (x'_i)\}$ is an equilibrium with exact market clearing. By definition $\sum_{i=1}^{I}(x'_i - e_i) = 0$, so markets clear exactly. Since $x'_i = x_i$ for $i \geq 2$, these agents maximize utility. Consider agent 1. By local nonsatiation we have $p \cdot x_i = p \cdot e_i$ for $i \geq 2$, so

$$p \cdot x'_1 = p \cdot \left(\sum_{i=1}^{I} e_i - \sum_{i=2}^{I} x_i \right) = p \cdot e_1 + \sum_{i=2}^{I} p \cdot (e_i - x_i) = p \cdot e_1.$$

Therefore agent 1 can afford the bundle x_1'. Since $\{p, (x_i)\}$ is an equilibrium, it must be $u_1(x_1') \le u_1(x_1)$. On the other hand, since u_1 is weakly monotonic and $x_1' \ge x_1$, we have $u_1(x_1') \ge u_1(x_1)$. Therefore $u_1(x_1') = u_1(x_1)$, so x_1' also maximizes utility.

3. (a) (2 points) Let $f(x) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}$, where $\gamma > 0$. Compute $-\frac{f''(x)}{f'(x)}$.

Solution: Since $f'(x) = \mathrm{e}^{-\gamma x}$ and $f''(x) = -\gamma \mathrm{e}^{-\gamma x}$, we have $-\frac{f''(x)}{f'(x)} = \gamma$.

(b) (3 points) Consider an economy with two goods that can be consumed in any amounts (positive or negative), and suppose that an agent has endowment $e = (a, b)$ and an additively separable utility function

$$U(x, y) = u(x) + v(y),$$

where $u' > 0$, $u'' < 0$, and similarly for v. Let $p_1 = p$ and $p_2 = 1$ be prices. Show that the agent's demand (x, y) satisfies

$$u'(x) - pv'(pa + b - px) = 0.$$

Solution: Since u, v are strictly increasing, the budget constraint must hold with equality. Since the initial wealth of the agent is $pa+b$, we pave $px+y = pa + b$, so $y = pa + b - px$. Therefore the utility maximization problem is equivalent to maximizing

$$u(x) + v(pa + b - px).$$

The first-order condition is

$$u'(x) - pv'(pa + b - px) = 0.$$

(c) (3 points) Regard x, the demand for good 1, as a function of the price p. Show that

$$\frac{\partial x}{\partial p} = \frac{v'(y) + pv''(y)(a - x)}{u''(x) + p^2 v''(y)},$$

where $y = pa + b - px$.

Solution: Let $F(x, p) = u'(x) - pv'(pa + b - px)$ be the left-hand side of the first-order condition. Then

$$\begin{aligned} F_x &= u''(x) + p^2 v''(pa + b - px) < 0, \\ F_p &= -v'(pa + b - px) - pv''(pa + b - px)(a - x). \end{aligned}$$

Hence by the implicit function theorem, we have

$$\frac{\partial x}{\partial p} = -\frac{F_p}{F_x} = \frac{v'(y) + pv''(y)(a-x)}{u''(x) + p^2 v''(y)},$$

where $y = pa + b - x$.

(d) (3 points) Show that

$$\frac{\partial x}{\partial p} = -\frac{1 - p\gamma_v(y)(a-x)}{p\gamma_u(x) + p^2\gamma_v(y)},$$

where $\gamma_u(x) = -\frac{u''(x)}{u'(x)}$ and similarly for v.

Solution: By the first-order condition, we have $u'(x) = pv'(y)$. Using the previous result and the definition of γ_u, γ_v, we obtain

$$\begin{aligned}\frac{\partial x}{\partial p} &= \frac{v'(y) + pv''(y)(a-x)}{u''(x) + p^2 v''(y)} = \frac{1 + p\frac{v''(y)}{v'(y)}(a-x)}{\frac{u'(x)}{v'(y)}\frac{u''(x)}{u'(x)} + p^2\frac{v''(y)}{v'(y)}} \\ &= \frac{1 - p\gamma_v(y)(a-x)}{-p\gamma_u(x) - p^2\gamma_v(y)} = -\frac{1 - p\gamma_v(y)(a-x)}{p\gamma_u(x) + p^2\gamma_v(y)}.\end{aligned}$$

(e) (3 points) Suppose that there are I agents indexed by $i = 1, \dots, I$. Agent i's endowment is $e_i = (a_i, b_i)$ and the utility function is

$$U_i(x, y) = -\frac{1}{\gamma_i}\left(\alpha_i \mathrm{e}^{-\gamma_i x} + (1-\alpha_i)\mathrm{e}^{-\gamma_i y}\right),$$

where $\gamma_i > 0$ and $0 < \alpha_i < 1$. Let x_i be agent i's demand for good 1. Show that

$$\frac{\partial x_i}{\partial p} = -\frac{1}{\gamma_i p(1+p)} + \frac{1}{1+p}(a_i - x_i).$$

Solution: Letting $u_i(x) = -\frac{\alpha_i}{\gamma_i}\mathrm{e}^{-\gamma_i x}$ and $v_i(y) = -\frac{1-\alpha_i}{\gamma_i}\mathrm{e}^{-\gamma_i y}$, it follows that

$$\frac{\partial x_i}{\partial p} = -\frac{1 - p\gamma_i(a_i - x_i)}{p\gamma_i + p^2\gamma_i} = -\frac{1}{\gamma_i p(1+p)} + \frac{1}{1+p}(a_i - x_i).$$

(f) (3 points) Let $z_1(p) = \sum_{i=1}^{I}(x_i - a_i)$ be the aggregate excess demand for good 1. Show that if p is an equilibrium price, then $z_1'(p) < 0$.

Solution: If p is an equilibrium price, by market clearing we have $\sum_{i=1}^{I}(x_i - a_i) = 0$. Summing up the above expression for $\frac{\partial x_i}{\partial p}$ over i, we obtain

$$z_1'(p) = \sum_{i=1}^{I}\left[-\frac{1}{\gamma_i p(1+p)} + \frac{1}{1+p}(a_i - x_i)\right] = -\frac{1}{p(1+p)}\sum_{i=1}^{I}\frac{1}{\gamma_i} < 0.$$

(g) (3 points) Show that the equilibrium is unique.

Solution: Suppose that there are two equilibrium prices $p_1 < p_2$. Since $z_1(p_j) = 0$ and $z_1'(p_j) < 0$ for $j = 1, 2$, we have $z_1(p_j + \epsilon) < 0$ and $z_1(p_j - \epsilon) > 0$ for small enough $\epsilon > 0$. By the intermediate value theorem, there exists $p^* \in (p_1 + \epsilon, p_2 - \epsilon)$ such that $z_1(p^*) = 0$. Furthermore, the excess demand curve must cross 0 from below at at least one point. Therefore we may assume $z_1'(p^*) \geq 0$, contradicting the previous result. Therefore the equilibrium is unique.

4. Consider an economy with two goods indexed by $l = 1, 2$. Suppose that there is a small country (so it doesn't affect world prices) with two agents indexed by $i = 1, 2$ and endowments $e_1 = (3/2, 1)$, $e_2 = (1, 3/2)$. All agents have utility function

$$u(x_1, x_2) = x_1 x_2.$$

Below, always normalize the price of good 1 to be $p_1 = 1$.

(a) (3 points) Compute the autarky equilibrium allocation and price.

Solution: By symmetry, it must be $p_1 = p_2 = 1$ and $x_1 = x_2 = (5/4, 5/4)$.

(b) (4 points) Suppose that the country opens up to trade, and the price of good 2 changes to $p_2 = 1/2$. Compute the free trade allocation and utility and determine who gains/loses from trade.

Solution: Let $q = 1/2$ be the domestic price of good 2. Using the Cobb-Douglas formula, the demand of agent 1 is

$$(x_{11}, x_{12}) = \left(\frac{1}{2}\frac{3/2+q}{1}, \frac{1}{2}\frac{3/2+q}{q}\right) = (1, 2),$$

and the utility is

$$u_1 = 1 \times 2 = 2 > \left(\frac{5}{4}\right)^2 = \frac{25}{16}.$$

Similarly, the demand of agent 2 is

$$(x_{21}, x_{22}) = \left(\frac{1}{2}\frac{1+(3/2)q}{1}, \frac{1}{2}\frac{1+(3/2)q}{q}\right) = \left(\frac{7}{8}, \frac{7}{4}\right),$$

and the utility is

$$u_2 = \frac{7}{8} \times \frac{7}{4} = \frac{49}{32} < \frac{50}{32} = \frac{25}{16}.$$

Therefore agent 1 gains from trade and agent 2 loses from trade.

(c) (4 points) Suppose that the government imposes a tariff of $\tau = 1/4$ on the import of good 2, and the domestic price of good 2 becomes $q = p_2 + \tau = 1/2 + 1/4 = 3/4$. Let T be the tax revenue from the tariff, and suppose that the government gives out the tariff revenue equally to agents (so each agent gets $T/2$). Express T using q, τ, T.

Solution: Since agents have identical homothetic utility, the total demand depends only on aggregate wealth. Since the aggregate endowment is $(5/2, 5/2)$ and the tariff revenue is T, the aggregate demand for good 2 is

$$\frac{1}{2}\frac{(5/2) + (5/2)q + T}{q}.$$

The import of good 2 is thus

$$\frac{(5/2)(1+q) + T}{2q} - \frac{5}{2}.$$

Since import is taxed at rate $\tau = 1/4$, we must have

$$T = \tau\left(\frac{(5/2)(1+q) + T}{2q} - \frac{5}{2}\right).$$

(d) (4 points) Solve for the new allocation and show that all agents gain from trade.

Solution: Setting $\tau = 1/4$ and $q = 3/4$, the solution to the above equation is $T = 1/8$. After some algebra, the new allocation is

$$(x_{11}, x_{12}) = \left(\frac{37}{32}, \frac{37}{24}\right),$$
$$(x_{21}, x_{22}) = \left(\frac{35}{32}, \frac{35}{24}\right).$$

Clearly agent 1 is better off than agent 2 since consumption have common denominators but the numerators are larger. Furthermore,

$$\frac{35}{32} \times \frac{35}{24} > \frac{5}{4} \times \frac{5}{4} \iff \frac{7^2}{8 \times 6} = \frac{49}{48} > 1,$$

so agent 2 gains from trade. Therefore all agents gain from trade.

(e) (5 points) Propose a better policy than the government's. Compute the allocation and utility under your suggested policy.

> **Solution:** Adopt free trade but introduce direct tax/subsidies to make the previous allocation just affordable, that is, determine t_i (tax on agent i) such that $p \cdot x_i^t = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i^t)$, where x_i^t is the consumption of agent i under the tariff regime. Since $p = (1, 1/2)$, $e_1 = (3/2, 1)$, and $x_1^t = (37/32, 37/24)$, it follows that $t_1 = 7/96$ and $t_2 = -7/96$. The agent 1's demand under this transfer is
>
> $$\begin{aligned}(x_{11}, x_{12}) &= \left(\frac{1}{2}\frac{37/32 + (1/2)(37/24)}{1}, \frac{1}{2}\frac{37/32 + (1/2)(37/24)}{1/2}\right) \\ &= \frac{37}{32}\left(\frac{5}{6}, \frac{5}{3}\right).\end{aligned}$$
>
> For agent 2, just change 37 to 35. To see that agent 1 gains from the free trade policy, note that
>
> $$\frac{5}{6} \times \frac{5}{3} > 1 \times \frac{4}{3} \iff 25 > 24,$$
>
> so free trade Pareto dominates the tariff regime.

5. Consider an economy with two periods, denoted by $t = 0, 1$, and three agents, denoted by $i = 1, 2, 3$. There are two states at $t = 1$, denoted by $s = 1, 2$. The two states occur with equal probability $\pi_1 = \pi_2 = 1/2$. Suppose that agent i's utility function is

$$U_i(x_0, x_1, x_2) = u_i(x_0) + \pi_1 u_i(x_1) + \pi_2 u_i(x_2),$$

where x_0, x_1, x_2 denote the consumption at $t = 0$ and states $s = 1, 2$, and the Bernoulli utility functions $u_i(x)$ are given by

$$\begin{aligned}u_1(x) &= -\frac{4}{x}, \\ u_2(x) &= -\frac{4}{x-2}, \\ u_3(x) &= -\frac{4}{x+2}.\end{aligned}$$

The initial endowments $e_i = (e_{i0}, e_{i1}, e_{i2})$ are given by

$$\begin{aligned}e_1 &= (3, 5, 1), \\ e_2 &= (3, 3, 3), \\ e_3 &= (2, 4, 2).\end{aligned}$$

(a) (2 points) What is the name of this type of utility functions?

Solution: Hyperbolic absolute risk aversion (HARA) utilities. Note that a HARA utility has a representation

$$u(x) = \frac{1}{a-1}(ax+b)^{1-\frac{1}{a}}.$$

The given functions correspond to $a = 1/2$ and $b = 0, -1, 1$.

(b) (2 points) For a given level of consumption, which agent is the most risk averse? Answer based on reasoning.

Solution: The absolute risk aversion (ARA) coefficient is given by

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax+b}.$$

Since $a > 0$ is common across agents, ARA is larger when b is smaller. Therefore agent 2 is the most risk averse.

(c) (6 points) Normalize the price of $t = 0$ good to be $p_0 = 1$. Compute the equilibrium state prices p_1, p_2.

Solution: Since agents have HARA utility with common a, by the argument in the lecture note, we can compute the equilibrium price through aggregation. Let e_s be the aggregate endowment in state s. Then

$$\begin{aligned} e_0 &= 3+3+2 = 8, \\ e_1 &= 5+3+4 = 12, \\ e_2 &= 1+3+2 = 6. \end{aligned}$$

Let $\pi_0 = 1$. Then the equilibrium price is given by

$$p_s = C\pi_s(ae_s + b)^{-\frac{1}{a}}$$

for some constant $C > 0$, where $b = \sum_{i=1}^{3} b_i = 0 + (-2) + 2 = 0$. Setting $a = 1/2$ and $b = 0$, it follows that

$$1 = p_0 = C(8/2)^{-2} = \iff C = 16,$$

so

$$\begin{aligned} p_1 &= 16\frac{1}{2}(12/2)^{-2} = \frac{2}{9}, \\ p_2 &= 16\frac{1}{2}(6/2)^{-2} = \frac{8}{9}. \end{aligned}$$

(d) (3 points) Compute the (gross) risk-free interest rate.

Solution: The risk-free asset pays 1 in states $s = 1, 2$. Therefore its price is

$$q = p_1 + p_2 = \frac{2}{9} + \frac{8}{9} = \frac{10}{9}.$$

The gross risk-free rate is

$$R_f = \frac{1}{q} = \frac{9}{10}.$$

(e) (3 points) Consider an asset (stock) that pays out the aggregate endowment as dividend. Compute the ex-dividend stock price (the stock price excluding the dividend) at $t = 0$.

Solution: The stock pays e_s in state s. Therefore its price is

$$q = p_1 e_1 + p_2 e_2 = \frac{2}{9} 12 + \frac{8}{9} 6 = 8.$$

(f) (2 points) Compute the expected stock return at $t = 0$ and show that it is higher than the risk-free rate.

Solution: Since each state occurs with probability 1/2, the expected stock return is

$$\mathrm{E}[R] = \frac{\mathrm{E}[e_s]}{q} = \frac{1}{8}\frac{1}{2}(12 + 6) = \frac{9}{8} > \frac{9}{10} = R_f.$$

(g) (2 points) Compute the price at $t = 0$ of a put option written on a stock with strike price 9.

Solution: Since the stock price (including the dividend) is e_s in state s, and $e_1 = 12 > 9$ and $e_2 = 6 < 9$, the put is exercised only in state 2. Letting $K = 9$ be the strike, the put price is therefore

$$q = p_2(K - e_2) = \frac{8}{9}(9 - 6) = \frac{8}{3}.$$

You can detach this sheet and use as a scratch paper.