# Econ 200B General Equilibrium
# Midterm Exam

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February 6, 2019

Name: ______________________________

Instructions:

- Read these instructions and the questions carefully. Questions are not necessarily ordered in the order of difficulty.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, i.e., base $e = 2.718281828\dots$.
- "Show" or "prove" require a formal mathematical proof or argument. "Explain" only requires an intuitive, though logically coherent, verbal argument.
- This exam has 4 questions on 8 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

| Question: | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | 10 | 20 | 30 | 40 | 100 |
| Score: | | | | | |

1. (10 points) As discussed during lectures, the equilibrium of an Arrow-Debreu economy need not be unique. Discuss conditions under which the equilibrium is unique. (No proofs are required but the more exhaustive the list is, the better.)

**Solution:** Examples of equilibrium uniqueness are:

- quasi-linear economy with strictly concave nonlinear parts,
- additively separable preferences with relative risk aversion bounded above by 1,
- identical homothetic preferences with strictly quasi-concave utility functions,
- arbitrary homothetic preferences with strictly quasi-concave utility functions and collinear endowments.

2. (20 points) Consider an Arrow-Debreu economy with two agents denoted by $i = A, B$ and $S$ states denoted by $s = 1, \dots, S$. Let $\pi_s > 0$ be the objective probability of state $s$, where $\sum_{s=1}^{S} \pi_s = 1$. Let $e_{is} > 0$ be agent $i$'s initial endowment of good $s$. Suppose that the utility functions are given by

$$U_A(x) = \sum_{s=1}^{S} \pi_s u(x_s),$$
$$U_B(x) = \sum_{s=1}^{S} \pi_s x_s,$$

where $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. What is the most you can say about equilibrium prices and allocations?

**Solution:** Since utilities are continuous, quasi-concave, and locally nonsatiated, an equilibrium exists. Let $p = (p_1, \dots, p_S)'$ be the price vector. The Lagrangian of the utility maximization problems are

$$L_A(x, \lambda_A) = \sum_{s=1}^{S} \pi_s u(x_s) + \lambda_A \left( w_A - \sum_{s=1}^{S} p_s x_s \right),$$
$$L_B(x, \lambda_B) = \sum_{s=1}^{S} \pi_s x_s + \lambda_B \left( w_B - \sum_{s=1}^{S} p_s x_s \right) + \sum_{s=1}^{S} \nu_s x_s,$$

where $w_A, w_B$ are initial wealth, $\lambda_A, \lambda_B \geq 0$ are Lagrange multipliers for the budget constraint, and $\nu_s \geq 0$ is the Lagrange multiplier for the nonnegativity constraint $x_s \geq 0$ for agent $B$. Note that we can ignore the nonnegativity constraint for agent $A$ due to the Inada condition.

Suppose agent $B$'s nonnegativity constraint does not bind in state $s$, so $x_{Bs} > 0$. By the first-order condition, we obtain $\pi_s - \lambda_B p_s = 0$. Since price levels are irrelevant for equilibrium, without loss of generality we may assume $\lambda_B = 1$ and hence $p_s = \pi_s$. Thus the price is equal to the state probability whenever agent $B$ consumes a positive amount. By the first-order condition for agent $A$, we then obtain

$$\pi_s u'(x_{As}) = \lambda_A p_s = \lambda_A \pi_s \iff u'(x_{As}) = \lambda_A.$$

Therefore $x_{As} = (u')^{-1}(\lambda_A)$ for all $s$ with $x_{Bs} > 0$, so agent $A$ consumes the same amount in every state. Thus the risk neutral agent $B$ accepts all the risk and provides perfect insurance to the risk averse agent $A$ for all states in which the nonnegativity constraint does not bind.

3. Consider an Arrow-Debreu economy with two agents indexed by $i = 1, 2$. Suppose that the utility functions are

$$\begin{aligned} U_1(x_1, x_2) &= \alpha \log x_1 + (1-\alpha) \log x_2, \\ U_2(x_1, x_2) &= \min\{x_1, x_2\}, \end{aligned}$$

where $0 < \alpha < 1$ is a preference parameter. Let agent $i$'s initial endowment be $(e_{i1}, e_{i2})$. Let $p_1 = 1$ and $p_2 = p$ be the prices.

(a) (10 points) Compute each agent's demand for good 1, given $p$.

**Solution:** Agent 1 has a Cobb-Douglas utility function, so by the usual formula the demand for good 1 is $x_{11} = \alpha w_1 = \alpha(e_{11} + pe_{12})$. Agent 2 has a Leontief utility function, so the demand must satisfy $x_1 = x_2$. Using the budget constraint, we obtain $x_{21} = \frac{e_{21}+pe_{22}}{1+p}$.

(b) (10 points) Derive a necessary and sufficient condition such that $p$ is an equilibrium price for some $e_{12} > 0$.

**Solution:** The excess demand function for good 1 is

$$\begin{aligned} z_1(p) &= x_{11} + x_{21} - e_{11} - e_{21} \\ &= \alpha(e_{11} + pe_{12}) + \frac{e_{21} + pe_{22}}{1+p} - e_{11} - e_{21} \\ &= -(1-\alpha)e_{11} + \alpha p e_{12} + \frac{p}{1+p}(e_{22} - e_{21}). \end{aligned}$$

Since Cobb-Douglas utility is strongly monotonic, by the Walras law a necessary and sufficient condition for equilibrium is $z_1(p) = 0$. Therefore $p$ is an equilibrium price for some $e_{12} > 0$ if and only if

$$\alpha p e_{12} = (1-\alpha)e_{11} + \frac{p}{1+p}(e_{21} - e_{22}) > 0.$$

(c) (10 points) Show that when agent 1 likes good 1 more ($\alpha$ increases), the relative price of good 1, which is $1/p$, increases (so $p$ decreases).

**Solution:** Let $F(p, \alpha)$ be the excess demand of good 1 given the price $p$ and the preference parameter $\alpha$. In equilibrium, we have $F(p, \alpha) = 0$. By the implicit function theorem, we have

$$\frac{\partial p}{\partial \alpha} = -\frac{\partial F}{\partial \alpha} \Big/ \frac{\partial F}{\partial p}.$$

Clearly

$$\frac{\partial F}{\partial \alpha} = e_{11} + pe_{12} > 0.$$

Therefore the slope of the relative price $1/p$ has the same sign as $\partial F/\partial p$. Using the above equilibrium condition, we obtain

$$\begin{aligned}
\frac{\partial F}{\partial p} &= \alpha e_{12} - \frac{e_{21} - e_{22}}{(1+p)^2} \\
&= \frac{(1-\alpha)e_{11}}{p} + \frac{e_{21} - e_{22}}{1+p} - \frac{e_{21} - e_{22}}{(1+p)^2} \\
&= \frac{1-\alpha}{p} e_{11} + \frac{p}{(1+p)^2}(e_{21} - e_{22}).
\end{aligned}$$

If $e_{22} \leq e_{21}$, clearly $\partial F/\partial p > 0$. If $e_{22} > e_{21}$, since $\frac{p}{1+p} < 1$, it follows that

$$\begin{aligned}
p\frac{\partial F}{\partial p} &= (1-\alpha)e_{11} + \left(\frac{p}{1+p}\right)^2 (e_{21} - e_{22}) \\
&> (1-\alpha)e_{11} + \frac{p}{1+p}(e_{21} - e_{22}) > 0
\end{aligned}$$

by the equilibrium condition. Therefore $\partial F/\partial p > 0$ at the equilibrium, so the relative price of good 1, $1/p$, always increases when agent 1 likes good 1 more.

4. Consider the following general equilibrium model. There are three time periods indexed by $t = 0, 1, 2$. There is a continuum of ex ante identical agents, where the population is normalized to 1. At $t = 0$, agents are endowed with $e > 0$ units of consumption good. At $t = 0$, agents can invest goods in two technologies. One unit of investment in technology 1 yields 1 unit of good at $t = 1$. One unit of investment in technology 2 yields $R > 0$ units of good at $t = 2$. Agents get utility only from consumption at $t = 1, 2$. At the beginning of $t = 1$, agents get "liquidity shocks", and with probability $\pi_i > 0$, their utility function becomes

$$U_i(x_1, x_2) = (1 - \beta_i) \log x_1 + \beta_i \log x_2,$$

where $\beta_i \in (0, 1)$ is the discount factor of type $i$ and $\sum_{i=1}^{I} \pi_i = 1$. Without loss of generality, assume

$$\beta_1 < \cdots < \beta_I,$$

so a type with a smaller index is more impatient. Suppose that the ex ante utility is

$$U((x_{i1}, x_{is})_i) = \sum_{i=1}^{I} \pi_i \alpha_i U_i(x_{i1}, x_{i2}),$$

where $\alpha_i > 0$ is the weight on type $i$ such that

$$\alpha_1 > \cdots > \alpha_I,$$

so agents care about emergencies in the sense that they put more utility weight on the impatient type. Note that we assume the law of large numbers, so at $t = 1$, exactly fraction $\pi_i > 0$ of agents are of type $i$. After observing their patience type at $t = 1$, agents can trade consumption for $t = 1, 2$ at a competitive (Arrow-Debreu) market.

(a) (10 points) In general, let $f, g$ be strictly increasing functions and $X$ be a random variable. Prove the Chebyshev inequality

$$E[f(X)g(X)] \geq E[f(X)]E[g(X)],$$

with equality if and only if $X$ is constant almost surely.
(Hint: let $X'$ be an i.i.d. copy of $X$ and consider the expectation of the quantity $(f(X) - f(X'))(g(X) - g(X')) \geq 0$.)

**Solution:** Since $f, g$ are strictly increasing, $f(X) - f(X')$ and $g(X) - g(X')$ have the same sign, and they are 0 if and only if $X = X'$. Therefore

$$(f(X) - f(X'))(g(X) - g(X')) \geq 0$$

with equality if and only if $X = X'$. Taking the expectation and noting that $X'$ is and i.i.d. copy of $X$, we obtain

$$E[f(X)g(X)] \geq E[f(X)]E[g(X)],$$

with equality if and only if $X$ is constant almost surely.

(b) (5 points) Noting that agents are ex ante identical, at $t = 0$ they will all make the same investment decision. let $x \in (0, e)$ be the amount of investment in technology 1 and let $(e_1, e_2) = (x, R(e - x))$ be the vector of $t = 1, 2$ endowments conditional on $x$. Let $(p_1, p_2) = (1, p)$ be the price of consumption at $t = 1, 2$. Compute type $i$'s demand for the $t = 1, 2$ goods using $p, e_1, e_2$.

**Solution:** The wealth of an agent at $t = 1$ is

$$w = p_1 e_1 + p_2 e_2 = e_1 + p e_2.$$

Since agents have Cobb-Douglas utility, using the familiar formula, type $i$'s demand is

$$x_{i1} = (1 - \beta_i)w = (1 - \beta_i)(e_1 + p e_2),$$
$$x_{i2} = \frac{\beta_i w}{p} = \beta_i \frac{e_1 + p e_2}{p}.$$

(c) (5 points) For notational simplicity, let $\bar{\beta} = \sum_{i=1}^{I} \pi_i \beta_i$ be the average discount factor. Conditional on $x$, compute the equilibrium price $p$.

> **Solution:** Since the population of agents is 1, the aggregate supply of good at $t = 1$ is $e_1 = x$. Noting that fraction $\pi_i$ of agents are of type $i$, the market clearing condition is
>
> $$\sum_{i=1}^{I} \pi_i (1 - \beta_i)(e_1 + p e_2) = x.$$
>
> Noting that $(e_1, e_2) = (x, R(e - x))$ and solving for $p$, we obtain
>
> $$p = \frac{1 - \sum_{i=1}^{I} \pi_i (1 - \beta_i)}{R(e - x) \sum_{i=1}^{I} \pi_i (1 - \beta_i)} x = \frac{\bar{\beta} x}{R(e - x)(1 - \bar{\beta})}.$$

(d) (10 points) Let $V(x, p) = \max U((x_{i1}, x_{i2})_i)$ be the agents' maximized utility conditional on short-term investment $x$ and price $p$. Noting that agents choose $x$ optimally given $p$, compute the equilibrium short-term investment $x^*$.

> **Solution:** Since markets are competitive, each agent maximizes $V(x, p)$ over $x$, taking $p$ as given. Since by the above demand formula $x_{i1}, x_{i2}$ are both proportional to $e_1 + p e_2 = x + pR(e - x)$, using
>
> $$V(x, p) = \sum_{i=1}^{I} \pi_i \alpha_i [(1 - \beta_i) \log x_{i1} + \beta_i \log x_{i2}],$$
>
> we obtain
>
> $$0 = \frac{\partial}{\partial x} V(x, p) = \sum_{i=1}^{I} \pi_i \alpha_i \frac{1 - pR}{x + pR(e - x)} \iff p = \frac{1}{R}.$$
>
> Using the above price formula, we obtain
>
> $$p = \frac{\bar{\beta} x}{R(e - x)(1 - \bar{\beta})} = \frac{1}{R} \iff x^* = (1 - \bar{\beta}) e.$$

(e) (10 points) Suppose that the government can force the agents to choose a particular $x$, without interfering in the subsequent consumption markets at $t = 1, 2$. Let $p(x)$ be the price of $t = 2$ consumption conditional on $x$ derived above. Prove that

$$\left. \frac{d}{dx} V(x, p(x)) \right|_{x = x^*} > 0,$$

so welfare locally increases if the government forces the agents to invest more in the short-term investment technology.

**Solution:** By the chain rule we have

$$\frac{d}{dx}V(x,p(x)) = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial p}p'(x).$$

At $x = x^*$, by the definition of equilibrium, we have $\partial V/\partial x = 0$. Since the numerator of $p(x)$ is increasing in $x$ and the denominator is decreasing in $x$, clearly $p$ is increasing in $x$, so $p'(x) > 0$. Therefore to show the claim, it suffices to show that $\partial V/\partial p > 0$ at $x = x^*$.

Since $x_{i1}$ is proportional to $e_1 + pe_2$ and $x_{i2}$ is proportional to $(e_1 + pe_2)/p$, by simple algebra we obtain

$$\frac{\partial V}{\partial p} = \sum_{i=1}^{I} \pi_i \alpha_i \left( \frac{e_2}{e_1 + pe_2} - \frac{\beta_i}{p} \right).$$

By the previous calculation, at $x = x^*$ we have $e_1 = x^* = (1 - \bar{\beta})e$, $e_2 = R(e - x^*) = R\bar{\beta}e$, and $p = 1/R$. Therefore we can simplify as

$$\frac{\partial V}{\partial p} = R\sum_{i=1}^{I} \pi_i \alpha_i (\bar{\beta} - \beta_i).$$

Since $\bar{\beta} = \sum_{i=1}^{I} \pi_i \beta_i$, the last summation can be interpreted as

$$E[\alpha_i]E[\beta_i] - E[\alpha_i \beta_i],$$

where the expectation is over the probabilities $(\pi_1, \dots, \pi_I)$. Since by assumption $\beta_i$ is increasing in $i$ and $\alpha_i$ is decreasing in $i$, applying the Chebyshev inequality to $-\alpha_i$ and $\beta_i$, we obtain

$$E[-\alpha_i \beta_i] > E[-\alpha_i]E[\beta_i] \iff E[\alpha_i]E[\beta_i] - E[\alpha_i \beta_i] > 0,$$

so $\partial V/\partial p > 0$.

This question is based on Geanakoplos and Walsh "Inefficient Liquidity Provision", Economic Theory, 66(1):213-233 (2018).

You can detach this sheet and use as a scratch paper.