Econ 200B General Equilibrium
Midterm Exam

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Name: ____

Instructions:

- Read these instructions and the questions carefully. Questions are not necessarily ordered in the order of difficulty.
- Don't start the exam until instructed.
- Turn off any electronic devices and put them in your bag.
- Don't put anything on your desk except the exam sheet, pens, pencils, eraser, and your ID card (*no* calculator). Failure to do so may be regarded as academic dishonesty.
- All logarithms are natural logarithms, i.e., base $\mathrm{e} = 2.718281828\dots$.
- "Show" or "prove" require a formal mathematical proof or argument. "Explain" only requires an intuitive, though logically coherent, verbal argument.
- This exam has 4 questions on 8 pages excluding the cover page, for a total of 100 points.
- Write the answer in the space below each question, unless otherwise stated in the question. If you don't have enough space you can use other parts of the exam sheet, but make sure to indicate where.
- You can detach the last empty page and use it as a scratch sheet.

Question:	1	2	3	4	Total
Points:	10	30	30	30	100
Score:					

1. The hyperbolic absolute risk aversion (HARA) utility function satisfies

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax+b},$$

where a, b are constants and we only consider the range of x such that $ax + b > 0$.

(a) (3 points) Derive the functional form of u (up to a monotonic transformation) when $a = 0$.

> **Solution:** $u(x) = -be^{-x/b}$, see lecture note for derivation.

(b) (3 points) Derive the functional form of u (up to a monotonic transformation) when $a = 1$.

> **Solution:** $u(x) = \log(x + b)$, see lecture note for derivation.

(c) (4 points) Derive the functional form of u (up to a monotonic transformation) when $a \neq 0, 1$.

> **Solution:** $u(x) = \frac{1}{a-1}(ax + b)^{1-1/a}$, see lecture note for derivation.

2. Consider an economy with two goods and two agents $i = A, B$. Both agents have utility function

$$u(x_1, x_2) = \min\{x_1, x_2\}.$$

The endowments are $e_A = (1, a)$ and $e_B = (1, b)$, where $a, b > 0$. Below, normalize prices such that $p_1 + p_2 = 1$.

(a) (5 points) What is the name of this utility function?

> **Solution:** Leontief.

(b) (5 points) Does this economy necessarily have an equilibrium? Explain.

> **Solution:** Yes. The utility functions are continuous, quasi-concave, and locally nonsatiated. The endowments are strictly positive. Hence by the existence theorem in the lecture note, there exists an equilibrium.

(c) (5 points) If an equilibrium exists, is it necessarily Pareto efficient? Explain.

> **Solution:** Yes. Since the utility functions are locally nonsatiated, by the first welfare theorem the equilibrium is efficient.

(d) (10 points) Compute all possible equilibrium prices (if they exist) and construct an equilibrium allocation for each price.

Solution: Let p_1, p_2 be the prices of goods, where we normalize $p_1 + p_2 = 1$. Let $p_1 = p$ and $p_2 = 1 - p$. Since the utility function is Leontief, agents want to consume the same amount of both goods. Hence if the endowment is $(1, a)$, the demand satisfies

$$(p_1 + p_2)x = p_1 + p_2 a \iff x_1^A = p + (1 - p)a.$$

The market clearing condition for good 1 is

$$x_1^A + x_1^B \leq 2 \iff 2p + (1 - p)(a + b) \leq 2,$$

with equality if $p > 0$. Similarly, the market clearing condition for good 2 is

$$x_2^A + x_2^B \leq a + b \iff 2p + (1 - p)(a + b) \leq a + b,$$

with equality if $1 - p > 0 \iff p < 1$. There are three cases.

1. If $a + b < 2$, then by the second inequality it must be $p = 0$. Therefore an equilibrium is

$$\left\{(p_1, p_2), (x_1^A, x_2^A), (x_1^B, x_2^B)\right\} = \{(0, 1), (a, a), (b, b)\}.$$

2. If $a + b = 2$, then any $p \in [0, 1]$ is an equilibrium price, in which case the unique equilibrium is

$$\begin{aligned}&\left\{(p_1, p_2), (x_1^A, x_2^A), (x_1^B, x_2^B)\right\} \\ &= \{(p, 1 - p), (p + (1 - p)a, p + (1 - p)a), (p + (1 - p)b, p + (1 - p)b)\}.\end{aligned}$$

3. If $a + b > 2$, then by the first inequality it must be $p = 1$. Therefore an equilibrium is

$$\left\{(p_1, p_2), (x_1^A, x_2^A), (x_1^B, x_2^B)\right\} = \{(0, 1), (1, 1), (1, 1)\}.$$

(e) (5 points) Suppose $a + b = 2$, $a > 1$, and $p_1 \in (0, 1)$. Show that if agent A destroys a small enough amount of endowment of good 2, he becomes strictly better off in equilibrium. Is this strange?

Solution: Agent A's utility when $a + b = 2$ and $p_1 = p \in (0, 1)$ is the equilibrium price is $U_A = p + (1 - p)a$. If A destroys $\epsilon > 0$ of his initial endowment, the new equilibrium price becomes $(p_1, p_2) = (0, 1)$ with utility $U_A' = a - \epsilon$. Since

$$U_A' > U_A \iff a - \epsilon > p + (1 - p)a \iff \epsilon < p(a - 1),$$

if $a > 1$ and $p > 0$, for small enough $\epsilon > 0$, destroying endowment increases utility. This is not strange because the competitive equilibrium assumes competitive behavior, so agents take prices as given. (They do not take into account the price impact by destroying endowments.)

3. Consider the following general equilibrium model. There is a continuum of agents, where the population is normalized to 1. There are two types of agents indexed by $i = 1, 2$. There are three time periods indexed by $t = 0, 1, 2$. At $t = 0$, agents are born and their types are realized. Agents consume only at $t = 1, 2$, and they have identical endowments $(e_1, e_2) = (a, b)$. Type i agents have utility function

$$U_i(x_1, x_2) = (1 - \beta_i)u(x_1) + \beta_i u(x_2),$$

where $\beta_i \in (0, 1)$ is the discount factor, x_t is consumption at t, and u is a von Neumann-Morgenstern utility function. There are no markets at $t = 0$, and hence agents cannot trade assets contingent on their type realization. At $t = 1$, the economy becomes a standard Arrow-Debreu economy. Let $\pi_i > 0$ be the probability of being type i, which is also the fraction of type i agents.

(a) (5 points) Define and compute the competitive equilibrium when $u(x) = \log x$.

Solution: A competitive equilibrium consists of an allocation $(x_{i1}, x_{i2})_{i=1,2}$ and price vector (p_1, p_2) such that agents maximize utility subject to the budget constraint and markets clear, so $\sum_{i=1}^2 \pi_i x_{it} = e_t$ for $t = 1, 2$.
Let the price vector be $(p_1, p_2) = (1, p)$. If $u(x) = \log x$, using the Cobb-Douglas formula, the demand is $x_{i1} = (1 - \beta_i)w$ and $x_{i2} = \beta_i w/p$, where $w = e_1 + pe_2 = a + pb$ is initial wealth. Hence market clearing for good 1 implies

$$\sum_{i=1}^{2} \pi_i(1 - \beta_i)(a + pb) = a \iff p = \frac{\sum_{i=1}^2 \pi_i \beta_i}{\sum_{i=1}^2 \pi_i(1 - \beta_i)} \frac{a}{b}.$$

Then after some algebra, the competitive equilibrium allocation is

$$(x_{i1}, x_{i2}) = \left(\frac{(1 - \beta_i)a}{\sum_{i=1}^2 \pi_i(1 - \beta_i)}, \frac{\beta_i b}{\sum_{i=1}^2 \pi_i \beta_i} \right).$$

(b) (10 points) Consider the social planner's problem, which is to maximize the ex ante welfare

$$\mathrm{E}[U_i(x_{i1}, x_{i2})] = \sum_{i=1}^{2} \pi_i U_i(x_{i1}, x_{i2})$$

subject to the feasibility constraint. Solve the planner's problem when $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$. (As usual, $u(x) = \log x$ when $\gamma = 1$.)

Solution: Since the utility function is additively separable, the planner's problem is equivalent to maximizing the weighted sum of utilities subject to feasibility constraint state-by-state. Consider $t = 2$. The problem is thus to maximize $\sum_{i=1}^2 \pi_i \beta_i u_i(x_{i2})$ subject to $\sum_{i=1}^2 \pi_i x_{i2} \le b$. For notational simplicity write x_i instead of x_{i2}. Then the Lagrangian is

$$L = \sum_{i=1}^{2} \pi_i \beta_i \frac{x_i^{1-\gamma}}{1 - \gamma} + \lambda \left(b - \sum_{i=1}^{2} \pi_i x_i \right).$$

The first-order condition is

$$\pi_i \beta_i x_i^{-\gamma} = \lambda \pi_i \iff x_i = (\beta_i/\lambda)^{1/\gamma}.$$

Using the feasibility constraint, we obtain

$$b = \sum_{i=1}^{2} \pi_i x_{i2} = \lambda^{-1/\gamma} \sum_{i=1}^{2} \pi_i \beta_i^{1/\gamma} \iff \lambda^{-1/\gamma} = \frac{b}{\sum_{i=1}^{2} \pi_i \beta_i^{1/\gamma}}.$$

Therefore $x_{i2} = \frac{\beta_i^{1/\gamma} b}{\sum_{i=1}^{2} \pi_i \beta_i^{1/\gamma}}$. We can solve for x_{i1} by interchanging $\beta_i, 1-\beta_i$ and a, b. Therefore the allocation in planner's problem is

$$(x_{i1}, x_{i2}) = \left(\frac{(1-\beta_i)^{1/\gamma} a}{\sum_{i=1}^{2} \pi_i (1-\beta_i)^{1/\gamma}}, \frac{\beta_i^{1/\gamma} b}{\sum_{i=1}^{2} \pi_i \beta_i^{1/\gamma}} \right).$$

(c) (5 points) Show that the allocations in competitive equilibrium and social planner's problem coincide if $u(x) = \log x$.

Solution: Trivial from the above calculations.

(d) (10 points) Suppose that $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma \neq 1$ and $\beta_1 \neq \beta_2$. Show that for generic choices of (a, b), the allocations in competitive equilibrium and social planner's problem do not coincide. More specifically, show that for any $a > 0$, there is exactly one value of $b > 0$ such that the two allocations coincide.

Solution: Suppose that the allocations in competitive equilibrium and planner's problem coincide. Let p be the equilibrium price of good 2. Then we must have

$$p = \frac{\beta_i}{1-\beta_i} \frac{u'(x_{i2})}{u'(x_{i1})} = \frac{\beta_i}{1-\beta_i} \left(\frac{x_{i2}}{x_{i1}} \right)^{-\gamma} = \left(c \frac{b}{a} \right)^{-\gamma},$$

where $c = \sum_i \pi_i (1-\beta_i)^{1/\gamma} / \sum_i \pi_i \beta_i^{1/\gamma}$. In equilibrium, agent's consumption must satisfy the budget constraint, so

$$a + pb = x_{i1} + p x_{i2} = \frac{a}{\sum_i \pi_i (1-\beta_i)^{1/\gamma}} ((1-\beta_i)^{1/\gamma} + p^{1-1/\gamma} \beta_i^{1/\gamma}).$$

Using the expression for p above, after some algebra we also have

$$a + pb = \frac{a}{\sum_i \pi_i (1-\beta_i)^{1/\gamma}} \left(\sum_i \pi_i (1-\beta_i)^{1/\gamma} + p^{1-1/\gamma} \sum_i \pi_i \beta_i^{1/\gamma} \right).$$

Therefore a necessary and sufficient condition for $a + pb = x_{i1} + p x_{i2}$ for $i = 1, 2$ is

$$(1-\beta_1)^{1/\gamma} + p^{1-1/\gamma} \beta_1^{1/\gamma} = (1-\beta_2)^{1/\gamma} + p^{1-1/\gamma} \beta_2^{1/\gamma}$$

$$\iff \left(c \frac{b}{a} \right)^{1-\gamma} = p^{1-1/\gamma} = \frac{(1-\beta_1)^{1/\gamma} - (1-\beta_2)^{1/\gamma}}{\beta_2^{1/\gamma} - \beta_1^{1/\gamma}}.$$

Because $\beta_1 \neq \beta_2$, the right-hand side is positive. Since $\gamma \neq 1$, the left-hand side is monotonic in b and its range is $(0, \infty)$. Therefore there exists a unique b that makes the above equation true. This shows that for generic choices of (a, b), the allocations in competitive equilibrium and planner's problem are distinct.

4. Consider the binomial option pricing model discussed in the lectures. Time is denoted by $t = 0, 1, \dots, T$. The gross risk-free rate is constant at $R > 0$. Each period, the stock can go up or down, so

$$S_{t+1} = \begin{cases} US_t & \text{if stock goes up,} \\ DS_t & \text{if stock goes down,} \end{cases}$$

where $U > R > D$. Suppose the initial stock price $S_0 > 0$ is given and consider an American put option with strike price K and expiration T. Recall that a put option is the right to sell a stock at the strike price, and "American" means that the option can be exercised any time until expiration.

(a) (4 points) Suppose for simplicity that $T = 1$. Let u, d stand for the up and down states and p_u, p_d be the state prices. Derive two equations that p_u, p_d satisfy.

Solution: Since the stock pays US_0 in the up state and DS_0 in the down state, its price must be $US_0 p_u + DS_0 p_d$. Therefore

$$S_0 = US_0 p_u + DS_0 p_d \iff Up_u + Dp_d = 1.$$

Similarly, accounting for the bond price, we obtain $1 = Rp_u + Rp_d$.

(b) (3 points) Compute p_u, p_d.

Solution: Expressing in matrix form, we obtain

$$\begin{aligned} &\begin{bmatrix} U & D \\ R & R \end{bmatrix} \begin{bmatrix} p_u \\ p_d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \iff &\begin{bmatrix} p_u \\ p_d \end{bmatrix} = \frac{1}{R(U-D)} \begin{bmatrix} R & -D \\ -R & U \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{R(U-D)} \begin{bmatrix} R-D \\ U-R \end{bmatrix} \end{aligned}$$

(c) (3 points) Compute the price of the American put option.

Solution: The put payoff is $K - S$ if exercised. Rational agents exercise options only if $K - S > 0$. Therefore the price of the American put option is

$$P = \max\left\{K - S_0, p_u P(u) + p_d P(d)\right\},$$

where

$$\begin{aligned} P(u) &= \max\left\{0, K - US_0\right\}, \\ P(d) &= \max\left\{0, K - DS_0\right\} \end{aligned}$$

are the put payoffs in states u, d.

(d) (10 points) Now suppose that the expiration date T is arbitrary. Prove that the put option price is decreasing in the current stock price S_0.

Solution: Let $p = \frac{R-D}{U-D}$ be the risk-neutral probability of state u. Let P_t be the put option price at t and $P_{t+1}(u), P_{t+1}(d)$ be the put option prices at $t+1$ when the stock goes up or down. Then

$$P_t = \max\left\{K - S_t, \frac{1}{R}(pP_{t+1}(u) + (1-p)P(d)_{t+1}))\right\}.$$

Since the terminal payoff $\max\{0, K - S_T\}$ is decreasing in the current stock price S_0 (because $S_T = U^n D^{T-n} S_0$ for some n), it follows from backward induction that P_0 is decreasing in S_0.

(e) (10 points) Prove that the put option price is convex in the current stock price S_0.

Solution: Noting that linear functions are convex and the maximum of two convex functions is convex, it follows from backward induction that P_0 is convex in S_0.

You can detach this sheet and use as a scratch paper.