Risk Aversion

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October 11, 2017

Abstract

This note explains the classic Arrow-Pratt measure of risk aversion. The standard reference is Pratt (1964) and Arrow (1965).

1 Measures of risk aversion

Consider an agent with expected utility function $\mathrm{E}[u(w)]$, where w is wealth and $u : \mathbb{R}_{++} \to \mathbb{R}$ is a utility function with $u' > 0$ (increasing) and $u'' < 0$ (concave).

Suppose that the investor has initial wealth w. Let ϵ be a small gamble, so $\mathrm{E}[\epsilon] = 0$. Consider the following two options.

1. The gamble enters the investor's wealth additively, so his expected utility is $\mathrm{E}[u(w + \epsilon)]$.
2. The investor does not hold the gamble but gives up $a > 0$, so his utility is $u(w - a)$.

When is the investor indifferent between these two options? Of course, the answer is when a satisfies

$$\mathrm{E}[u(w + \epsilon)] = u(w - a). \tag{1}$$

Noting that ϵ is a small gamble, we can Taylor-expand the left-hand side of (1) to the second order:

$$\mathrm{E}[u(w + \epsilon)] \approx \mathrm{E}\left[u(w) + u'(w)\epsilon + \frac{1}{2}u''(w)\epsilon^2\right] = u(w) + \frac{1}{2}u''(w)\,\mathrm{E}[\epsilon^2], \tag{2}$$

where I used $\mathrm{E}[\epsilon] = 0$. (The reason why I expanded to the second order is because the first order term is zero.) Similarly, Taylor-expanding the right-hand side to the first order, we get

$$u(w - a) \approx u(w) - u'(w)a. \tag{3}$$

Combining (1)–(3) and replacing $\approx$ by $=$, we get

$$\begin{aligned}\mathrm{E}[u(w + \epsilon)] = u(w - a) &\iff u(w) + \frac{1}{2}u''(w)\,\mathrm{E}[\epsilon^2] = u(w) - u'(w)a \\ &\iff a = -\frac{u''(w)}{u'(w)}\frac{\mathrm{E}[\epsilon^2]}{2}.\end{aligned}$$

The last equation shows that the investor should give up an amount proportional to the variance in order to avoid risk. The term

$$\mathrm{ARA}(w) = -\frac{u''(w)}{u'(w)}$$

is called the *absolute risk aversion coefficient* at wealth w.

Now consider the same problem, except that everything is multiplicative. So the options are

1. the investor's expected utility is $\mathrm{E}[u(w(1+\epsilon))]$.
2. The investor does not hold the gamble but gives up a fraction $a > 0$ of his wealth, so his utility is $u(w(1-a))$.

By repeating the above calculation, we get

$$a = -\frac{wu''(w)}{u'(w)} \frac{\mathrm{E}[\epsilon^2]}{2}.$$

The term

$$\mathrm{RRA}(w) = -\frac{wu''(w)}{u'(w)}$$

is called the *relative risk aversion (RRA) coefficient* at wealth w.

2 CARA, CRRA, HARA utilities

In applications, it is often convenient to consider utility functions that have constant risk aversion coefficients, or other simple forms.

2.1 CARA

The constant absolute risk aversion (CARA) utility function satisfies

$$-\frac{u''(w)}{u'(w)} = \alpha$$

for some $\alpha > 0$. Putting a minus sign and integrating once, we get

$$\log u'(w) = -\alpha w + c,$$

where c is an arbitrary constant. Taking exponentials, we get

$$u'(w) = C\mathrm{e}^{-\alpha w},$$

where $C = \mathrm{e}^c > 0$. Integrating once again, we get

$$u(w) = -\frac{C}{\alpha}\mathrm{e}^{-\alpha w} + D,$$

where $C > 0$ and $D \in \mathbb{R}$ are arbitrary. Since $x \mapsto Cx + D$ is an affine (hence monotone) transformation, it does not matter for the behavior (utility maximization). Therefore we set $C = 1$ and $D = 0$, and the CARA utility function is

$$u(w) = -\frac{1}{\alpha}\mathrm{e}^{-\alpha w}.$$

2.2 CRRA

The constant relative risk aversion (CRRA) utility function satisfies

$$-\frac{wu''(w)}{u'(w)} = \gamma$$

for some $\gamma > 0$. By repeating the above calculation, we get the CRRA utility function

$$u(w) = \begin{cases} \frac{w^{1-\gamma}}{1-\gamma}, & (\gamma \neq 1) \\ \log w. & (\gamma = 1) \end{cases}$$

Many applied works use CRRA utility function, because it is the only additively separable utility function that is consistent with constant interest rate under balanced growth (*i.e.*, the interest rate does not change when everything (consumption, etc.) grows at a constant rate). If you look at aggregate consumption data, its logarithm behaves like a random walk, so consumption has roughly grown at a constant rate. However, the interest rate has not changed much over time, so it suggests CRRA is a good description.

What is a "reasonable" number for risk aversion? Suppose that you have to choose between a lifetime income of 1, or a lifetime income of u with probability $1/2$ and d with probability $1/2$, where $u > 1 > d > 0$ (u, d stand for up and down). If your preference is CRRA with risk aversion γ, you would be indifferent between the two options if

$$\frac{1}{2}\frac{u^{1-\gamma}}{1-\gamma} + \frac{1}{2}\frac{d^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \iff u = (2 - d^{1-\gamma})^{\frac{1}{1-\gamma}}.$$

Figure 1 below plots the implied value of u for various γ when $d = 0.9$, so there is a 50% chance that your lifetime income will decrease by 10% (and 50% chance that your lifetime income will increase by $100(u-1)$%). I think most people have risk aversion around 2–5. In their famous paper, Mehra and Prescott (1985) suggest that $\gamma = 10$ is an upper bound for a reasonable value of relative risk aversion.

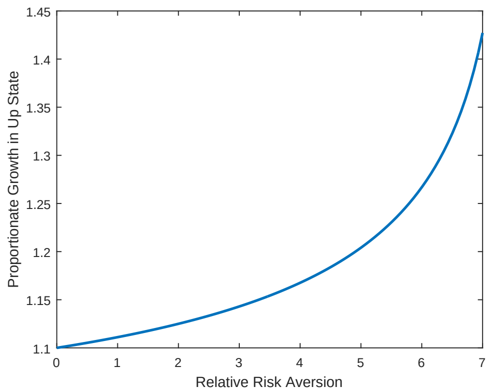

Figure 1. Implied value of u when $d = 0.9$.

2.3 HARA

The reciprocal of the absolute risk aversion coefficient, $-u'(w)/u''(w)$, is called the *risk tolerance*. A utility function whose risk tolerance is linear in wealth is called *linear risk tolerance (LRT)*. So LRT satisfies

$$-\frac{u'(w)}{u''(w)} = aw + b$$

for some constants a, b and w such that $aw+b > 0$ (in order to be consistent with $u' > 0$ and $u'' < 0$). LRT utility functions are also called *hyperbolic absolute risk aversion (HARA)*, because the graph of the absolute risk aversion coefficient is a hyperbola:

$$-\frac{u''(w)}{u'(w)} = \frac{1}{aw + b}.$$

By repeating the above calculation, we get

$$u(w) = \begin{cases} \frac{1}{a-1}(aw+b)^{1-1/a}, & (a \neq 0, 1) \\ -b\mathrm{e}^{-w/b}, & (a = 0) \\ \log(w+b). & (a = 1) \end{cases}$$

In particular, setting $a = -1$ (so we consider wealth levels of only $-w + b > 0 \iff w < b$), we get

$$u(w) = -\frac{1}{2}(b-w)^2,$$

so the quadratic utility function (as well as the CARA and CRRA utility functions) is a special case of HARA.

3 Portfolio choice with two assets

Consider an investor with expected utility of wealth $\mathrm{E}[u(w)]$. Suppose that there are two assets, a risky asset with gross return $R \geq 0$ (which is a random variable) and a risk-free asset with risk-free rate $R_f > 0$. Let θ be the fraction of wealth invested in the risky asset. Then the return on the portfolio θ is

$$R(\theta) := R\theta + R_f(1-\theta).$$

The optimal portfolio problem is

$$\max_\theta \mathrm{E}[u(R(\theta)w)].$$

Since we are considering a general utility function, we cannot solve for the optimal portfolio θ. However, we can make a few predictions about θ and derive comparative statics.

The following lemma is basic.

Lemma 1. *Consider an agent with initial wealth w and utility function u, where $u' > 0$ and $u'' < 0$. Let θ be the optimal portfolio. Then the following is true.*

1. *θ is unique.*

2. *$\theta \gtreqless 0$ according as* $\mathrm{E}[R] \gtreqless R_f$.

3. *Suppose* $\mathrm{E}[R] > R_f$. *If u exhibits decreasing relative risk aversion (DRRA), so $-xu''(x)/u'(x)$ is decreasing, then $\partial\theta/\partial w \geq 0$,* i.e., *the agent invests comparatively more in the risky asset as he becomes richer. The opposite is true if u exhibits increasing relative risk aversion (IRRA).*

Proof. 1. Let $f(\theta) = \mathrm{E}[u(R(\theta)w)]$. Then $f'(\theta) = \mathrm{E}[u'(R(\theta)w)(R - R_f)w]$ and $f''(\theta) = \mathrm{E}[u''(R(\theta)w)(R - R_f)^2 w^2] < 0$, so f is strictly concave. Therefore the optimal θ is unique (if it exists).

2. Since $f'(\theta) = 0$ and $f'(0) = u'(R_f w)w(\mathrm{E}[R] - R_f)$, the result follows.

3. Dividing the first-order condition by w, we obtain $\mathrm{E}[u'(R(\theta)w)(R - R_f)] = 0$. Let $F(\theta, w)$ be the left-hand side. Then by the implicit function theorem we have $\partial\theta/\partial w = -F_w/F_\theta$. Since $F_\theta = \mathrm{E}[u''(R(\theta)w)(R - R_f)^2 w] < 0$, it suffices to show $F_w \geq 0$. Let $\gamma(x) = -xu''(x)/u'(x) > 0$ be the relative risk aversion coefficient. Then

$$F_w = \mathrm{E}[u''(R(\theta)w)(R - R_f)R(\theta)] = -\frac{1}{w}\,\mathrm{E}[\gamma(R(\theta)w)u'(R(\theta)w)(R - R_f)].$$

Since $\mathrm{E}[R] > R_f$, by the previous result we have $\theta > 0$. Therefore

$$R(\theta) = R\theta + R_f(1 - \theta) \gtreqless R_f$$

according as $R \gtreqless R_f$. Since u is DRRA, γ is decreasing, so $\gamma(R(\theta)w) \leq \gamma(R_f w)$ if $R \geq R_f$ (and reverse inequality if $R \leq R_f$). Therefore

$$\gamma(R(\theta)w)(R - R_f) \leq \gamma(R_f w)(R - R_f)$$

always. Multiplying both sides by $-u'(R(\theta)w) < 0$ and taking expectations, we obtain

$$\begin{aligned}
F_w &= -\frac{1}{w}\,\mathrm{E}[\gamma(R(\theta)w)u'(R(\theta)w)(R - R_f)] \\
&\geq -\frac{1}{w}\,\mathrm{E}[\gamma(R_f w)u'(R(\theta)w)(R - R_f)] = 0,
\end{aligned}$$

where the last equality uses the first-order condition. □

Lemma 2. *Suppose* $\mathrm{E}[R] > R_f$ *and consider two agents indexed by $i = 1, 2$. Let w_i, $u_i(x)$, $\gamma_i(x) = -xu_i''(x)/u_i'(x)$, and θ_i be the initial wealth, utility function, relative risk aversion, and the optimal portfolio of agent i. If $\gamma_1(w_1 x) > \gamma_2(w_2 x)$ for all x, then $\theta_1 < \theta_2$,* i.e., *the more risk averse agent invests comparatively less in the risky asset.*

Proof. Since $\gamma_1(w_1 x) > \gamma_2(w_2 x)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_2'(w_2 x)}{u_1'(w_1 x)}\right) = \frac{w_2 u_2'' u_1' - u_2' w_1 u_1''}{(u_1')^2} = \frac{1}{x}\frac{u_2'}{u_1'}(\gamma_1(w_1 x) - \gamma_2(w_2 x)) > 0,$$

so $u_2'(w_2 x)/u_1'(w_1)$ is increasing. Letting $\theta = \theta_1 > 0$ (which follows from $\mathrm{E}[R] > R_f$), since $R(\theta) \gtreqless R_f$ according as $R \gtreqless R_f$ and $u_2'(w_2 x)/u_1'(w_1 x)$ is increasing (and positive), we have

$$\frac{u_2'(R(\theta)w_2)}{u_1'(R(\theta)w_1)}(R - R_f) > \frac{u_2'(R_f w_2)}{u_1'(R_f w_1)}(R - R_f)$$

always (except when $R = R_f$). Multiplying both sides by $u_1'(R(\theta)w_1) > 0$ and taking expectations, we get

$$\begin{aligned}
\mathrm{E}[u_2'(R(\theta)w_2)(R - R_f)] &= \mathrm{E}\left[\frac{u_2'(R(\theta)w_2)}{u_1'(R(\theta)w_1)} u_1'(R(\theta)w_1)(R - R_f)\right] \\
&> \mathrm{E}\left[\frac{u_2'(R_f w_2)}{u_1'(R_f w_1)} u_1'(R(\theta)w_1)(R - R_f)\right] = 0,
\end{aligned}$$

where the last equality uses the first-order condition for agent 1. Therefore $\theta_2 > \theta = \theta_1$. □

Lemma 3. *Consider an agent with initial wealth w, utility function $u(x)$ and relative risk aversion $\gamma(x) = -xu''(x)/u'(x)$. Let R_f be the risk-free rate and θ the optimal portfolio. If (i) $\gamma(x) \leq 1$, or (ii) u is DARA and $\theta \geq 1$, then $\partial\theta/\partial R_f < 0$.*

Proof. By the first-order condition, we have $\mathrm{E}[u'(R(\theta)w)(R - R_f)] = 0$. Let $F(\theta, R_f)$ be the left-hand side. Then by the implicit function theorem we have $\partial\theta/\partial R_f = -F_{R_f}/F_\theta$. Since $F_\theta = \mathrm{E}[u''(R(\theta)w)(R - R_f)^2 w] < 0$, it suffices to show $F_{R_f} < 0$. Differentiating F with respect to R_f and using the definition of the relative risk aversion, we get

$$F_{R_f} = \mathrm{E}[u''(R(\theta)w)(1 - \theta)(R - R_f)w] - \mathrm{E}[u'(R(\theta)w)]. \tag{4}$$

Case 1: $\gamma(x) \leq 1$. By (4), we obtain

$$\begin{aligned}
F_{R_f} &= \mathrm{E}\left[-\frac{\gamma(R(\theta)w)}{R(\theta)} u'(R(\theta)w)(1 - \theta)(R - R_f)\right] - \mathrm{E}[u'(R(\theta)w)] \\
&= -\mathrm{E}\left[\frac{u'(R(\theta)w)}{R(\theta)} \left(\gamma(1 - \theta)(R - R_f) + R(\theta)\right)\right].
\end{aligned}$$

Since $(1 - \theta)(R - R_f) = R - R\theta - R_f(1 - \theta) = R - R(\theta)$, it follows that

$$\begin{aligned}
F_{R_f} &= -\mathrm{E}\left[\frac{u'(R(\theta)w)}{R(\theta)} \left(\gamma R - \gamma R(\theta) + R(\theta)\right)\right] \\
&= -\mathrm{E}\left[\frac{u'(R(\theta)w)}{R(\theta)} \left(\gamma R + (1 - \gamma)R(\theta)\right)\right].
\end{aligned}$$

Since $u', R, R(\theta) > 0$, if $\gamma \leq 1$, then $F_{R_f} < 0$.

Case 2: u is DARA and $\theta \geq 1$. Letting $a(x) = -u''(x)/u'(x)$ be the absolute risk aversion, it follows from (4) that

$$F_{R_f} = -(1 - \theta)w\, \mathrm{E}[a(R(\theta)w)u'(R(\theta)w)(R - R_f)] - \mathrm{E}[u'(R(\theta)w)].$$

Since $\theta \geq 1$ and $u' > 0$, it suffices to show that

$$\mathrm{E}[a(R(\theta)w)u'(R(\theta)w)(R - R_f)] \leq 0.$$

But since u is DARA, $a(\cdot)$ is decreasing, so we have $a(R(\theta)w) \lesseqgtr a(R_f w)$ according as $R \gtreqless R_f$. Hence $a(R(\theta)w)(R - R_f) \leq a(R_f w)(R - R_f)$ always, so

$$\begin{aligned}
\mathrm{E}[a(R(\theta)w)u'(R(\theta)w)(R - R_f)] &\leq \mathrm{E}[a(R_f w)u'(R(\theta)w)(R - R_f)] \\
&= a(R_f w)\, \mathrm{E}[u'(R(\theta)w)(R - R_f)] = 0,
\end{aligned}$$

where we have used $u' > 0$ and the first-order condition. □

Some of the above results can be found in Toda and Walsh (2014).

Exercises

1. Derive the relative risk aversion coefficient by imitating the derivation of the absolute risk aversion coefficient.

2. Derive the CRRA utility function by imitating the derivation of the CARA utility function. Make sure to treat the cases $\gamma = 1$ and $\gamma \neq 1$ separately.

3. Derive the HARA utility function by imitating the derivation of the CARA utility function. Make sure to treat the cases $a = 0$, $a = 1$, and $a \neq 0, 1$ separately.

4. In Section 3 we defined a portfolio by the *fraction of wealth* θ invested in the risky asset. Instead, let $x = w\theta$ be the *amount of wealth* invested in the risky asset. Then the final wealth is $w_1(x) = Rx + R_f(w - x)$, where w is initial wealth. By emulating Lemma 1, show that if $\mathrm{E}[R] > R_f$ and u exhibits decreasing absolute risk aversion (DARA), then $\partial x/\partial w \geq 0$, *i.e.*, the agent invests more amount in the risky asset as he becomes richer.

5. Formulate and prove a version of Lemma 2 for invested amount x instead of fraction of investment θ.

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