Stochastic Calculus and Control

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Abstract

This note explains the basics of stochastic calculus and control. More details can be found in Chang (2004), Shreve (2004), and references therein. The formal theory of stochastic calculus is hard, but its intuitive understanding and applications are manageable.

1 Brownian motion

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The discrete-time stochastic process $\{X(t)\}_{t=0}^{\infty}$ is called a *random walk* if $X(0) = 0$ and $X(t) = X(t-1) + \epsilon_t$, where $\{\epsilon_t\}_{t=1}^{\infty}$ is i.i.d. with mean 0 and variance σ^2. Let $0 = t_0 < t_1 < t_2 < \cdots < t_N$. We can easily show that the random variables $\{X(t_n) - X(t_{n-1})\}_{n=1}^{N}$ are independent, mean zero, and

$$\mathrm{Var}[X(t_n) - X(t_{n-1})] = \sigma^2(t_n - t_{n-1}).$$

Now divide the time interval $[t, t+1)$ into subintervals with equal length Δt, where $1/\Delta t$ is the number of subintervals. In order to define $X(t)$ on the boundary points of these intervals, it is natural to define

$$X(t) = \sum_{k=1}^{t/\Delta t} u_k,$$

where $\{u_k\}_{k=0}^{\infty}$ is i.i.d. with mean zero. Since $\sigma^2 t = \mathrm{Var}[X(t)] = (t/\Delta t)\,\mathrm{Var}[u_k]$ at integer t, we must have $\mathrm{Var}[u_k] = \sigma^2 \Delta t$. In this case, for $0 = t_0 < t_1 < \cdots < t_N$, it is still true that the random variables $\{X(t_n) - X(t_{n-1})\}_{n=1}^{N}$ are independent, mean zero, and

$$\mathrm{Var}[X(t_n) - X(t_{n-1})] = \sigma^2(t_n - t_{n-1}).$$

Furthermore, by letting $\Delta t \to 0$, since there will be more and more u_k's between time $t > s$, by the central limit theorem we have

$$X(t) - X(s) \sim N(0, \sigma^2(t-s)).$$

Formally, the continuous-time stochastic process $\{X(t)\}_{t \geq 0}$ is a *Brownian motion* if

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1. $X(0) = 0$,
2. the sample path $t \mapsto X(t)$ is continuous almost surely, and
3. for $0 < t_1 < \cdots < t_N$, the random variables $\{X(t_n) - X(t_{n-1})\}_{n=1}^N$ are independent, and $X(t) - X(s) \sim N(0, \sigma^2(t-s))$ for $t > s$.

When $\sigma^2 = 1$ we say that $X(t)$ is standard Brownian motion, which is often denoted by $B(t)$, B_t, $W(t)$, or W_t.[1]

We can also consider the multidimensional Brownian motion. $X : [0, \infty) \times \Omega \to \mathbb{R}^d$ is a d-dimensional Brownian motion if $X(0) = 0$, $t \mapsto X(t)$ has continuous sample paths, and X has independent increments with

$$X(t) - X(s) \sim N(0, (t-s)\Sigma),$$

where Σ is a positive semidefinite instantaneous variance matrix.

2 Stochastic calculus

2.1 Stochastic integral

Now that we defined the Brownian motion, we want to do calculus (integration and differentiation) of functions of the Brownian motion. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration, that is, an increasing sequence of σ-algebra. Assume $a : [0, \infty) \times \Omega$ and B is adapted to $\{\mathcal{F}_t\}$, so $a(t, \cdot)$ and $B(t, \cdot)$ are $\mathcal{F}_t$-measurable. First, we make sense of the integral

$$\int_0^T a(t, \omega) \, \mathrm{d}B(t).$$

If a is constant in time, so $a(t, \omega) = a(\omega)$, it is natural to define

$$\int_0^T a(\omega) \, \mathrm{d}B(t) = a(\omega) B(T).$$

If a is a step function in time, so there exist $0 = t_0 < t_1 < \cdots < t_N = T$ such that $a(t, \omega)$ is constant on each interval $[t_{n-1}, t_n)$, by carrying out the integral over each subinterval, it is natural to define

$$\int_0^T a(t, \omega) \, \mathrm{d}B(t) = \sum_{n=1}^N a(t_{n-1}, \omega)(B(t_n) - B(t_{n-1})).$$

In general, assume that $t \mapsto a(t, \omega)$ has sample paths that are continuous from the right (so $\lim_{s \downarrow t} a(s, \omega) = a(t, \omega)$) and have limits from the left (so $\lim_{s \uparrow t} a(s, \omega)$ exists). Such functions are called càdlàg, which is the abbreviation of the French sentence "continue à droite, limite à gauche" (meaning "continuous at right, limit at left"). If a is càdlàg, it is natural to define

$$\int_0^T a(t, \omega) \, \mathrm{d}B(t) = \lim \sum_{n=1}^N a(t_{n-1}, \omega)(B(t_n) - B(t_{n-1})), \tag{1}$$

[1] Since Norbert Wiener developed the mathematical theory of the Brownian motion, it is also called the *Wiener process*, hence the notation $W(t)$. In economics and finance, the letter W is often reserved for wealth, so B is more common.

where the limit is taken over all partitions $0 = t_0 < t_1 < \cdots < t_N = T$ that become finer and finer. Using a similar argument to the definition of the Riemann integral, we can show that this definition makes sense for functions that are square integrable with finite variance,

$$\mathrm{E}\left[\int_0^T |a(t,\omega)|^2 \,\mathrm{d}t\right] < \infty.$$

The definition of the stochastic integral (also known as the Itô integral) (1) is very much similar to the definition of the Riemann-Stieltjes integral

$$\int_a^b f(x)\,\mathrm{d}g(x) = \lim \sum_{n=1}^N f(\xi_n)(g(x_n) - g(x_{n-1})), \tag{2}$$

where $a = x_0 < x_1 < \cdots < x_N = b$ and ξ_n is a point in the interval $[x_{n-1}, x_n]$. The only difference is that in the definition of the stochastic integral (1), the integrand $a(t,\omega)$ is evaluated at the left of the interval, t_{n-1}. On the other hand, in the definition of the Riemann-Stieltjes integral (2), the integrand $f(x)$ is evaluated at an arbitrary point $\xi_n \in [x_{n-1}, x_n]$. The reason is that the limit (1) will depend on the choice of the evaluation point.[2] In economics and finance, since agents act upon current information, it is natural to use the left point, for otherwise the agent must know the future in order to calculate the stochastic integral.

For example, let us compute $\int_0^T B\,\mathrm{d}B$. Divide $[0,T]$ into N equally spaced intervals, so the boundary points are $t_n = nT/N$ for $n = 0, 1, \dots, N$. For notational simplicity let $B(t_n) = B_n$. Then

$$\begin{aligned}\int_0^T B\,\mathrm{d}B &\approx \sum_{n=1}^N B(t_{n-1})(B(t_n) - B(t_{n-1})) \\ &= \sum_{n=1}^N B_{n-1}(B_n - B_{n-1}) \\ &= \sum_{n=1}^N \frac{1}{2}\left(B_n^2 - B_{n-1}^2 - (B_n - B_{n-1})^2\right) \\ &= \frac{1}{2}B(T)^2 - \frac{1}{2}\frac{1}{N}\sum_{n=1}^N (\sqrt{N}(B_n - B_{n-1}))^2.\end{aligned}$$

By the definition of the Brownian motion, the random variables $\left\{\sqrt{N}(B_n - B_{n-1})\right\}_{n=1}^N$ are independent, zero mean, and have variance $N(t_n - t_{n-1}) = NT/N = T$. Therefore by the law of large numbers, as $N \to \infty$ we have

$$\frac{1}{N}\sum_{n=1}^N (\sqrt{N}(B_n - B_{n-1}))^2 \xrightarrow{\text{a.s.}} T.$$

[2] The integral using the mid point $(t_n + t_{n-1})/2$ is called the Stratonovich integral, but we will never use it.

Therefore letting $N \to \infty$, we get

$$\int_0^T B \, \mathrm{d}B = \frac{1}{2}B(T)^2 - \frac{1}{2}T.$$

In stochastic calculus, it makes our life easy to work with differentials like $\mathrm{d}t$ and $\mathrm{d}B$ instead of integrals. The rules to remember are $(\mathrm{d}t)^2 = 0$, $\mathrm{d}t \, \mathrm{d}B = 0$, and $(\mathrm{d}B)^2 = \mathrm{d}t$. The justifications are

$$\sum_{n=1}^N (t_n - t_{n-1})^2 \to 0, \tag{3a}$$

$$\sum_{n=1}^N (t_n - t_{n-1})(B(t_n) - B(t_{n-1})) \xrightarrow{\text{a.s.}} 0, \tag{3b}$$

$$\sum_{n=1}^N (B(t_n) - B(t_{n-1}))^2 \xrightarrow{\text{a.s.}} T = \int_0^T \mathrm{d}t, \tag{3c}$$

respectively.

We can define stochastic integrals with respect to other stochastic processes. For example, suppose that $X(t)$ satisfies

$$X(t) = X(0) + \int_0^t g(s,\omega) \, \mathrm{d}s + \int_0^t v(s,\omega) \, \mathrm{d}B(s)$$

for all $t \geq 0$, where g, v are adapted processes. Such a process is called a *diffusion* (or an Itô process), and is compactly written as

$$\mathrm{d}X(t) = g(t,\omega) \, \mathrm{d}t + v(t,\omega) \, \mathrm{d}B(t).$$

g, v are called the *drift* and the *diffusion* coefficients. Intuitively, g, v are the instantaneous growth and volatility (hence the letters g, v)). Using differentials for the Brownian motion, we can justify the notations $\mathrm{d}t \, \mathrm{d}X = 0$ and $(\mathrm{d}X)^2 = v^2 \, \mathrm{d}t$, for example.

Then we can define the stochastic integral with respect to X by

$$\int_0^T a(t,\omega) \, \mathrm{d}X(t) = \int_0^T a(t,\omega) g(t,\omega) \, \mathrm{d}t + \int_0^T a(t,\omega) v(t,\omega) \, \mathrm{d}B(t).$$

If X is multidimensional, then the stochastic integral is defined element-by-element.

2.2 Itô's formula

In ordinary calculus, we almost never use the definition of the Riemann integral for computation. Instead, we use the fundamental theorem of calculus,

$$\int_a^b f'(x) \, \mathrm{d}x = f(b) - f(a),$$

which holds if f is C^1 (or more generally, absolutely continuous: see Folland (1999)). We can symbolically write $\mathrm{d}f(x) = f'(x) \, \mathrm{d}x$. Itô's formula is just the stochastic version of this.

Let $\{X(t)\}_{t\geq 0}$ be a stochastic process with continuous sample paths, for example Brownian motion. Let f be a C^2 function. Let $0 < t_1 < t_2 < \cdots < t_N = t$ and $X(t_n) = X_n$. Then by Taylor's theorem,

$$f(X(t)) - f(X(0)) = \sum_{n=1}^{N}(f(X(t_n)) - f(X(t_{n-1})))$$
$$= \sum_{n=1}^{N}\left[f'(X_{n-1})(X_n - X_{n-1}) + \frac{1}{2}f''((1-\theta_n)X_{n-1} + \theta_n X_n)(X_n - X_{n-1})^2\right],$$

where $\theta_n \in [0,1]$. If $X(t) = B(t)$ is Brownian motion, letting $N \to \infty$ the first term converges to the stochastic integral

$$\int_0^T f'(B)\,\mathrm{d}B.$$

By the continuity of f'' and a similar argument as in the calculation of $\int_0^T B\,\mathrm{d}B$ above, the second term converges to the Riemann integral

$$\int_0^T \frac{1}{2}f''(B)\,\mathrm{d}t.$$

Therefore

$$f(B(t)) - f(B(0)) = \int_0^T f'(B)\,\mathrm{d}B + \int_0^T \frac{1}{2}f''(B)\,\mathrm{d}t,$$

which we write symbolically as

$$\mathrm{d}f(B) = f'(B)\,\mathrm{d}B + \frac{1}{2}f''(B)\,\mathrm{d}t. \tag{4}$$

(4) is known as *Itô's formula*, or Itô's lemma.[3]

If $f = f(t,x)$ also depends on time directly, by a similar argument we get

$$f(B(t)) - f(B(0)) = \int_0^T f_t(t,B)\,\mathrm{d}t + \int_0^T f_x(t,B)\,\mathrm{d}B + \int_0^T \frac{1}{2}f_{xx}(t,B)\,\mathrm{d}t,$$

which we write as

$$\mathrm{d}f(t,B) = f_t\,\mathrm{d}t + f_x\,\mathrm{d}B + \frac{1}{2}f_{xx}\,\mathrm{d}t.$$

An easier way to remember might be

$$\mathrm{d}f(t,B) = f_t\,\mathrm{d}t + f_x\,\mathrm{d}B + \frac{1}{2}f_{xx}(\mathrm{d}B)^2,$$

since $(\mathrm{d}B)^2 = \mathrm{d}t$.

If $f = f(t,x)$ and $X(t)$ is a general diffusion satisfying

$$\mathrm{d}X = g\,\mathrm{d}t + v\,\mathrm{d}B,$$

[3]There is a mathematical joke that all important theorems are called lemmas. Examples are Itô's lemma and Zorn's lemma.

then by a similar argument we can show

$$\begin{aligned}
\mathrm{d}f(t,X) &= f_t\,\mathrm{d}t + f_x\,\mathrm{d}X + \frac{1}{2}f_{xx}(\mathrm{d}X)^2 \\
&= f_t\,\mathrm{d}t + f_x(g\,\mathrm{d}t + v\,\mathrm{d}B) + \frac{1}{2}f_{xx}v^2\,\mathrm{d}t \\
&= \left(f_t + gf_x + \frac{1}{2}v^2 f_{xx}\right)\mathrm{d}t + vf_x\,\mathrm{d}B.
\end{aligned}$$

Therefore if X is a diffusion and f is a C^2 function, then $f(t,X)$ is also a diffusion.

We can also consider the multidimensional Itô's formula. If

$$\mathrm{d}X = g\,\mathrm{d}t + V\,\mathrm{d}B,$$

where g is $d_x \times 1$, V is $d_x \times d_b$, and B is a d_b-dimensional Brownian motion with instantaneous variance Σ, then by analogy we get

$$\begin{aligned}
\mathrm{d}f(t,X) &= f_t\,\mathrm{d}t + f_x\,\mathrm{d}X + \frac{1}{2}(\mathrm{d}X)'f_{xx}(\mathrm{d}X) \\
&= f_t\,\mathrm{d}t + f_x(g\,\mathrm{d}t + V\,\mathrm{d}B) + \frac{1}{2}(V\,\mathrm{d}B)'f_{xx}(V\,\mathrm{d}B) \\
&= f_t\,\mathrm{d}t + f_x(g\,\mathrm{d}t + V\,\mathrm{d}B) + \frac{1}{2}(\mathrm{d}B)'V'f_{xx}V(\mathrm{d}B) \\
&= f_t\,\mathrm{d}t + f_x(g\,\mathrm{d}t + V\,\mathrm{d}B) + \frac{1}{2}\operatorname{tr}[(\mathrm{d}B)'(V'f_{xx}V)(\mathrm{d}B)] \\
&= f_t\,\mathrm{d}t + f_x(g\,\mathrm{d}t + V\,\mathrm{d}B) + \frac{1}{2}\operatorname{tr}[f_{xx}V(\mathrm{d}B)(\mathrm{d}B)'V'] \\
&= f_t\,\mathrm{d}t + f_x(g\,\mathrm{d}t + V\,\mathrm{d}B) + \frac{1}{2}\operatorname{tr}[f_{xx}(V\Sigma V')]\,\mathrm{d}t \\
&= \left(f_t + f_x g + \frac{1}{2}\operatorname{tr}[f_{xx}(V\Sigma V')]\right)\mathrm{d}t + f_x V\,\mathrm{d}B.
\end{aligned}$$

Don't try to remember these formulas. The only things you need to do to recover these formulas are

1. Taylor expand the function $f(t,x)$ to the second order, and
2. use the rules $(\mathrm{d}t)^2 = 0$, $\mathrm{d}t\,\mathrm{d}B = 0$, and $(\mathrm{d}B)(\mathrm{d}B)' = \Sigma\,\mathrm{d}t$.

3 Stochastic control

3.1 Hamilton-Jacobi-Bellman equation

In basic calculus, after learning differentiation, we apply it to optimization. The same holds for stochastic calculus. Consider the problem

$$\begin{aligned}
&\text{maximize} && \mathrm{E}_0\left[\int_0^\infty f(s,X(s),Y(s))\,\mathrm{d}s\right] \\
&\text{subject to} && \mathrm{d}X(t) = g(t,X(t),Y(t))\,\mathrm{d}t + V(t,X(t),Y(t))\,\mathrm{d}B, \\
& && X(0)\text{ given,}
\end{aligned}$$

where f is continuous, C^1 in t, and C^2 in x, $X(t)$ is the state variable, $Y(t)$ is the control variable, and B is a multidimensional Brownian motion with instantaneous variance Σ.

In discrete-time, we solve such dynamic programming problems by deriving the Bellman equation. The same is true for continuous-time. Let

$$J(t,x)=\sup_{\{Y(s)\}_{s\geq t}}\mathrm{E}_t\left[\int_t^\infty f(s,X(s),Y(s))\,\mathrm{d}s\right]$$

be the value function from time t on, *i.e.*, the maximum continuation value at time t when the state is $X(t)=x$. If Δt is small, by the Bellman equation for the discrete-time, we get

$$\begin{aligned}J(t,x)&\approx\sup_y\left\{f(t,x,y)\Delta t+\mathrm{E}_t[J(t+\Delta t,x+\Delta x)]\right\}\\ \iff 0&\approx\sup_y\left\{f(t,x,y)\Delta t+\mathrm{E}_t[\Delta J]\right\},\end{aligned}$$

where $\Delta J=J(t+\Delta t,x+\Delta x)-J(t,x)$. Dividing both sides by $\Delta t>0$ and letting $\Delta t\to 0$, we get

$$0=\sup_y\left\{f(t,x,y)+\mathrm{E}_t[\mathrm{d}J]/\,\mathrm{d}t\right\}.$$

Of course, we need to make sense of $\mathrm{E}_t[\mathrm{d}J]/\,\mathrm{d}t$. By Itô's formula, we get

$$\mathrm{d}J=\left(J_t+J_xg+\frac{1}{2}\operatorname{tr}[J_{xx}(V\Sigma V')]\right)\mathrm{d}t+J_xV\,\mathrm{d}B.$$

Taking expectations and noting that the increment of Brownian motion is normal (hence zero mean), we get

$$\mathrm{E}_t[\mathrm{d}J]=\left(J_t+J_xg+\frac{1}{2}\operatorname{tr}[J_{xx}(V\Sigma V')]\right)\mathrm{d}t.$$

Therefore the Bellman equation reduces to

$$0=\sup_y\left\{f(t,x,y)+J_t+J_xg+\frac{1}{2}\operatorname{tr}[J_{xx}(V\Sigma V')]\right\}. \tag{5}$$

(5) is known as the *Hamilton-Jacobi-Bellman (HJB) equation*.

(5) is particularly useful for finding the solution. The plan $\{(X(t),Y(t))\}_{t\geq 0}$ is called *feasible* if it satisfies the constraint. We can prove the following theorem.

Theorem 1. *Let $J(t,x)$ be C^1 in t and C^2 in x, $\{(X(t),Y(t))\}_{t\geq 0}$ be feasible, and $\mathrm{E}_0\left[\int_0^\infty |f(s,X(s),Y(s))|\,\mathrm{d}s\right]<\infty$. If (i) $J(t,x)$ satisfies the HJB equation* (5)*, and (ii) the transversality condition*

$$\lim_{T\to\infty}\mathrm{E}_t[J(T,X(T))]=0$$

holds, then

$$\mathrm{E}_t\left[\int_t^\infty f(s,X(s),Y(s))\,\mathrm{d}s\right]\leq J_t(X(t)).$$

Equality holds if $y=Y(t)$ is the arg max *of the HJB equation at $x=X(t)$.*

Proof. By the Bellman equation, we get

$$f(s,X(s),Y(s)) \le -\left(J_t + J_x g + \frac{1}{2}\operatorname{tr}[J_{xx}(V\Sigma V')]\right).$$

Integrating from $s=t$ to $s=T$, we get

$$\int_t^T f(s,X(s),Y(s))\,\mathrm{d}s \le -\int_t^T \left(J_t + J_x g + \frac{1}{2}\operatorname{tr}[J_{xx}(V\Sigma V')]\right)\mathrm{d}s.$$

Taking conditional expectations, noting that the Brownian motion has zero mean increments (*i.e.*, it is a martingale), and using Itô's formula, we get

$$\begin{aligned}
&\mathrm{E}_t\left[\int_t^T f(s,X(s),Y(s))\,\mathrm{d}s\right]\\
&\le -\mathrm{E}_t\left[\int_t^T \left(J_t + J_x g + \frac{1}{2}\operatorname{tr}[J_{xx}(V\Sigma V')]\right)\mathrm{d}s\right]\\
&= -\mathrm{E}_t\left[\int_t^T \left(J_t + J_x g + \frac{1}{2}\operatorname{tr}[J_{xx}(V\Sigma V')]\right)\mathrm{d}s + J_x g\,\mathrm{d}B\right]\\
&= J(t,X(t)) - \mathrm{E}_t[J(T,X(T))].
\end{aligned}$$

Letting $T\to\infty$ and invoking the Dominated Convergence Theorem and the transversality condition, we get

$$\mathrm{E}_t\left[\int_t^\infty f(s,X(s),Y(s))\,\mathrm{d}s\right] \le J_t(X(t)). \quad \square$$

In summary, if the value function is smooth, the HJB equation is necessary. Conversely, HJB equation and transversality condition are sufficient for optimality.

Remark. In many applications, utilities are discounted, so $f(t,x,y) = \mathrm{e}^{-\beta t}u(x,y)$. In this case, letting $\tilde{J}(x) = \mathrm{e}^{\beta t}J(t,x)$ be the undiscounted value function, the HJB equation (5) reduces to

$$0 = \max_y \left\{u(x,y) - \beta\tilde{J} + \tilde{J}_x g + \frac{1}{2}\operatorname{tr}[\tilde{J}_{xx}(V\Sigma V')]\right\}. \tag{6}$$

3.2 Merton (1971)

As an application of stochastic control, consider the classic Merton (1971) optimal consumption-portfolio problem. In this model, the investor has flow utility $u(c_t)$ from consumption and can invest in a risk-free asset (with interest rate r) and risky assets indexed by $k=1,\dots,K$. The price of asset k evolves according to the diffusion

$$\mathrm{d}P_k/P_k = \mu_k\,\mathrm{d}t + \sigma_k\,\mathrm{d}B_k,$$

where μ_k is the expected return, σ_k is the volatility, and B_k is a standard Brownian motion (it may be correlated across assets). Letting w_t be the wealth of agent at time t and θ_t be the portfolio, the budget constraint is

$$\mathrm{d}w_t = ((r + (\mu - r\mathbf{1})'\theta_t)w_t - c_t)\,\mathrm{d}t + \sum_{k=1}^J \theta_k w_t \sigma_k\,\mathrm{d}B_k.$$

The state variable is wealth w and the control variables are consumption c and portfolio θ. The HJB equation in undiscounted form is

$$0 = \max_{c,\theta} \left\{ u(c) - \beta J + J'(w)[(r + (\mu - r\mathbf{1})'\theta)w - c] + \frac{1}{2} w^2 J''(w)\theta'\Sigma\theta \right\},$$

where Σ is the instantaneous variance of the Brownian motion. The first-order condition with respect to c is

$$u'(c) = J'(w).$$

The first-order condition with respect to θ is

$$wJ'(w)(\mu - r\mathbf{1}) + w^2 J''(w)\Sigma\theta = 0 \iff \theta = -\frac{J'(w)}{wJ''(w)}\Sigma^{-1}(\mu - r\mathbf{1}).$$

If $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, by homotheticity we can guess $J(w) = A\frac{w^{1-\gamma}}{1-\gamma}$ for some $A > 0$, and the optimal portfolio is

$$\theta = \frac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{1}).$$

Substituting the optimal portfolio into the HJB equation, we can also solve for the optimal consumption rule, which is $c_t = mw_t$ with

$$m = \beta\varepsilon + (1 - \varepsilon)\left(r + \frac{1}{2\gamma}(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1}) \right), \tag{7}$$

where $\varepsilon = 1/\gamma$ is the elasticity of intertemporal substitution. See Merton (1969, 1971) for more details and Svensson (1989) for the case with Epstein-Zin preference (the formula is the same). For recursive utility in continuous-time, see Duffie and Epstein (1992a,b).

3.3 Saito (1998)

A useful trick to obtain heterogeneous-agent general equilibrium models that are analytically tractable is to use the optimal portfolio problem as the building block. The simplest of such models is Saito (1998).[4]

There are two technologies indexed by $j = 1, 2$. Capital invested in technology 1 (stock market) evolves according to the geometric Brownian motion

$$\mathrm{d}K_t/K_t = \mu_1 \,\mathrm{d}t + \sigma_1 \,\mathrm{d}B_{1t}.$$

Capital invested in technology 2 (private equity, human capital, etc.) is subject to aggregate and idiosyncratic risks and evolves according to

$$\mathrm{d}K_t/K_t = \mu_2 \,\mathrm{d}t + \sigma_2 \,\mathrm{d}B_{2t} + \sigma_i \,\mathrm{d}B_{it}.$$

[4]Other interesting papers of this type are Angeletos and Panousi (2011), Benhabib et al. (2011); **?** (continuous-time) and Constantinides and Duffie (1996), Krebs (2003a,b, 2006, 2007), Toda (2014, 2015) (discrete-time), which use the CRRA preference with multiplicative shocks. It is also possible to use the CARA preference with additive shocks. See Calvet (2001), Angeletos and Calvet (2005, 2006), and Wang (2007).

Here B_{1t}, B_{2t} are standard Brownian motions satisfying $\mathrm{d}B_1\,\mathrm{d}B_2 = \rho\,\mathrm{d}t$, where $-1 \le \rho \le 1$ is the correlation coefficient, and B_{it} is a standard Brownian motion for agent i, which is i.i.d. across agents and independent from aggregate shocks B_1, B_2. Each agent maximizes the the CRRA utility

$$\mathrm{E}_0\left[\int_0^\infty \mathrm{e}^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}\,\mathrm{d}t\right].$$

subject to the budget constraint

$$\begin{aligned}\mathrm{d}w_t = (1-\theta_1-\theta_2)rw_t\,\mathrm{d}t + \theta_1(\mu_1\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_{1t})w_t \\ + \theta_2(\mu_2\,\mathrm{d}t + \sigma_2\,\mathrm{d}B_{2t} + \sigma_i\,\mathrm{d}B_{it})w_t - c_t\,\mathrm{d}t,\end{aligned}$$

where r is the (equilibrium) risk-free rate and $\theta = (\theta_1, \theta_2)$ is the portfolio of technologies. This problem is a standard Merton (1971) type optimal consumption-portfolio problem except that I use recursive utility, which has been solved by Svensson (1989). Letting

$$\mu = \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2+\sigma_i^2\end{bmatrix},$$

the optimal portfolio is

$$\theta = \frac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{1})$$

and the optimal consumption rule is $c_t = mw_t$, where the marginal propensity to consume out of wealth is

$$m = \beta\varepsilon + (1-\varepsilon)\left(r + \frac{1}{2\gamma}(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})\right),$$

where $\varepsilon = 1/\gamma$ is EIS.

So far, it's just a straightforward partial equilibrium application of Merton. To make it general equilibrium, note that the portfolio choice is the same for every agent. Since the risk-free asset is in zero net supply, in equilibrium nobody holds the risk-free asset. The market clearing condition is therefore

$$\begin{aligned}1-\theta_1-\theta_2 = 0 &\iff \mathbf{1}'\frac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{1}) = 1 \\ &\iff r = \frac{\mathbf{1}'\Sigma^{-1}\mu - \gamma}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}.\end{aligned}$$

You can see that the idiosyncratic shock generically affects the risk-free rate and the risk premia (which I discuss in detail in Toda (2015)). For example, assume $\mu_1 = \mu_2 = \mu$, so the expected return on the stock market and the private equity is the same. Then the risk-free rate becomes (after some algebra)

$$r = \mu - \frac{\gamma}{\mathbf{1}\Sigma^{-1}\mathbf{1}} = \mu - \gamma((1-\rho^2)\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_i^2)(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 + \sigma_i^2).$$

The equity premium is

$$\gamma((1-\rho^2)\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_i^2)(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 + \sigma_i^2) > 0$$

for both assets, and it is increasing in the idiosyncratic volatility σ_i.

Exercises

1. Prove (3a) and (3b). ((3c) is already proved in the text.)

2. Prove (6).

3. Derive (7).

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