Applied General Equilibrium Theory

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Contents

Chapter 1

Arrow-Debreu model

1.1 What is general equilibrium?

The term *general equilibrium* (GE for short) is the antonym of *partial equilibrium* (PE for short). Partial equilibrium is what we learn in introductory microeconomics, for example the demand curve, supply curve, consumer surplus, etc. In partial equilibrium, we assume that the demand and supply of a good depends only on the price of the good, so the markets for different goods are taken as independent. But in reality, markets are interdependent. A typical example is oil. Since oil is used in the production of many goods and services, when the price of oil changes, so do prices of other goods. For example, when the oil price goes up, so does the price of flight tickets. Similarly, the demand for SUVs (sport utility vehicles) may decrease and the demand for EVs (electric vehicles) may increase. As opposed to partial equilibrium analysis, general equilibrium theory takes into account the interactions of all markets.

By definition, a general equilibrium model features multiple goods and services (commodities). In order to increase the scope of applicability of general equilibrium theory, we need to use our imagination and interpret goods broadly: goods are distinguished not only by their physical properties (apples or bananas) but also by time (think how happy you will be to get a house in La Jolla either now or in 100 years), location (sushi in the middle of a desert is very different from sushi on the coast), and states (an umbrella on a sunny day is very different from one on a rainy day). Depending on which feature (time, location, uncertainty) of goods we focus on, general equilibrium theory can be applied to many different fields in economics. Table 1.1 shows what happens when time, location, and uncertainty are put into a general equilibrium model. For example, when a general equilibrium model features time, it is called macroeconomics (because macroeconomics is mainly concerned with economic fluctuations and growth over time). When it features location and uncertainty, it is called international finance. As we can see from Table 1.1, general equilibrium theory is the foundation of modern economics.

Table 1.1: Features in a general equilibrium model.

	Time	Location	Uncertainty
Time	Macro	International Macro	Macro-Finance
Location		Trade	International Finance
Uncertainty			Finance

1.2 Arrow-Debreu model defined

Let's introduce the Arrow-Debreu model of general equilibrium, named after Kenneth Arrow[1] and Gérard Debreu[2] for their work in the first mathematical proof of equilibrium existence.

There are I agents (consumers) indexed by $i \in I = \{1, 2, \dots, I\}$.[3] Agents are endowed with some goods. There are L goods indexed by $l \in L = \{1, 2, \dots, L\}$. There can also be firms who produce output goods from input goods, but let's ignore them for now.

Notations A *consumption bundle* is a vector[4] $x = (x_1, \dots, x_L) \in \mathbb{R}^L_+$ that specifies the quantity of each good. The *endowment* of agent i, denoted by $e_i = (e_{i1}, \dots, e_{iL}) \in \mathbb{R}^L_+$, is a particular bundle. Below are standard notations.

$$\begin{aligned}
&\mathbb{R}^L_+ = \left\{x = (x_1, \dots, x_L) \in \mathbb{R}^L \,\middle|\, x_l \ge 0 \text{ for all } l\right\},\\
&\mathbb{R}^L_{++} = \left\{x = (x_1, \dots, x_L) \in \mathbb{R}^L \,\middle|\, x_l > 0 \text{ for all } l\right\},\\
&x \ge y \iff x_l \ge y_l \text{ for all } l,\\
&x > y \iff x_l \ge y_l \text{ for all } l \text{ and } x_l > y_l \text{ for some } l,\\
&x \gg y \iff x_l > y_l \text{ for all } l.
\end{aligned}$$

The symbol $\forall$ and $\exists$ stand for "for all ...such that ..." and "there exists ...such that ...". So we often write $\forall$ and $\exists$ instead of spelling out a sentence, since it saves time and (after getting used to) it facilitates understanding.

Preference relation Each agent is assumed to have a well-defined preference over all possible consumption bundles. This is of course unrealistic—for instance it is hard for me to determine whether I prefer an olive or a pickle—but is a mathematical simplification. Agent i's preference relation is denoted by $\succsim_i$. $x \succsim_i y$ denotes that agent i weakly prefers the bundle x to y. Obviously, $x \succ_i y$ ($x \sim_i y$) means that agent i strictly prefers x to y (is indifferent between x and y). $x \succsim_i y$ is the same as $y \precsim_i x$.

By a "well-defined preference", I mean that for any bundles $x, y \in \mathbb{R}^L_+$, we have either $x \succsim_i y$ or $x \precsim_i y$ (or both, in which case $x \sim_i y$). This property is

[1] http://en.wikipedia.org/wiki/Kenneth_Arrow

[2] http://en.wikipedia.org/wiki/Gerard_Debreu

[3] Using the upper case letter for a finite set as well as its cardinality (here I) and a lower case letter for a typical element (here i) makes the notation simple and easy to remember. It is known as the Cass convention, after David Cass. http://en.wikipedia.org/wiki/David_Cass

[4] In this course all vectors are column vectors, like $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Since column vectors take up a lot of space, column vectors are written as $x = (x_1, x_2)'$, the transpose of a row vector, with the sign $'$. Even this is tiresome and I usually omit the transpose sign $'$.

called *completeness*. We assume that agents are logical in that if one prefers x to y and y to z, then he prefers x to z. Mathematically, this means that

$$x \succsim_i y, y \succsim_i z \implies x \succsim_i z.$$

This property is called *transitivity*. If you know a lot of math, you will find that a preference relation is nothing but a binary relation that is complete and transitive.

Utility function A function $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ is called a *utility function* if

$$x \succsim_i y \iff u_i(x) \geq u_i(y),$$

that is, agent i prefers bundle x to y if and only if x gives higher utility than y. If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then obviously

$$u_i(x) \geq u_i(y) \iff f(u_i(x)) \geq f(u_i(y)).$$

Therefore, if u_i is a utility function and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $v_i(x) = f(u_i(x))$ is also a utility function. So if there is a utility function, there are infinitely many utility functions because we can apply any monotonic transformation like $f(x) = 2x, x^3, \mathrm{e}^x$, etc. For this reason, it makes no sense to discuss the numerical (cardinal) value of utility. All that matters is the *order* among the utility of bundles. A property preserved by a monotonic transformation (i.e., applying a strictly increasing function) is called an *ordinal* property.

Here are a few examples of utility functions that will be used in the course (two good case).

Cobb-Douglas $u(x_1, x_2) = \alpha_1 \log x_1 + \alpha_2 \log x_2$. (Also $u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$.)

Leontief $u(x_1, x_2) = \min\{x_1/\alpha_1, x_2/\alpha_2\}$.

CES $u(x_1, x_2) = (\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma})^{\frac{1}{1-\sigma}}$. (Also $u(x_1, x_2) = \frac{1}{1-\sigma}(\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma})$.) CES stands for constant elasticity of substitution.

In both cases, α_1, α_2 are positive numbers. The generalization to multiple goods is obvious.

Given a preference relation, does a utility function always exist? The answer is no in general, but it is yes under weak and reasonable assumptions. The statement and the proof can be found in Debreu (1959, pp. 55–59), so I will not go into details and simply assume that a utility function exists.

Now we can define the economy.

Definition 1.1. An *Arrow-Debreu economy*

$$\mathcal{E} = \{I, (e_i), (u_i)\}$$

consists of the set of agents $I = \{1, 2, \ldots, I\}$, their endowments $(e_i) \subset \mathbb{R}^L_+$, and their utility functions $u_i : \mathbb{R}^L_+ \to \mathbb{R}$.

Note that $e_i = (e_{i1}, \ldots, e_{iL})'$ is the endowment vector of agent i.

Markets, prices, and budgets We assume that there is a market for each good. Such a model is called a *complete market* model. In the real world, of course, not all goods are tradable. For instance, you cannot sell your future labor income, because if you can you might decide not to work and run away after selling. Such a model is called an *incomplete market* model and plays an important role in models with time and uncertainty (especially in finance), but we will not treat that case in this course since it's substantially more complicated.

Good l is quoted by a price $p_l \geq 0$ in some unit of account. The price can be zero, in which case it is called a free good, like air. Again in the real world, there are goods (bads) that have a negative price, like garbage (which you have to *pay* to get rid of), a junk car, or nuclear waste. We abstract from that aspect and assume *free disposal*—that if you have goods that you don't want to consume, you can dispose of them for free. Therefore all prices are nonnegative. The vector $p = (p_1, \dots, p_L) \in \mathbb{R}^L_+$ is called the *price vector*, or simply the price.

Each agent faces a budget. If agent i is endowed with $e_i \in \mathbb{R}^L_+$, then he has e_{il} of good l. If he sells at price p_l, he gets $p_l e_{il}$ units of account. Therefore by selling all of his endowment, he gets wealth

$$w_i = \sum_{l=1}^{L} p_l e_{il} = p \cdot e_i,$$

which is the inner product of price p and endowment e_i. The inner product is also denoted by $\langle p, e_i \rangle$ (common notation in mathematics) or $p' e_i$ (common notation in econometrics and statistics).

If agent i wishes to consume the bundle $x = (x_1, \dots, x_L)' \in \mathbb{R}^L_+$, he must spend $p_l x_l$ units of account on good l. Therefore his total expenditure is

$$\sum_{l=1}^{L} p_l x_l = p \cdot x,$$

again an inner product. A bundle x is said to be *affordable*, or *budget feasible*, if

$$p \cdot x \leq p \cdot e_i \iff p \cdot (x - e_i) \leq 0,$$

that is, total expenditure is less than or equal to wealth. The set of all affordable bundles for agent i,

$$B_i(p) = \left\{ x \in \mathbb{R}^L_+ \,\middle|\, p \cdot (x - e_i) \leq 0 \right\},$$

is called the *budget set* (Figure 1.1). Note that if $t > 0$, then $B_i(tp) = B_i(p)$, so the budget set is *homogeneous of degree 0* in price. In words, it says that it does not matter how we quote the price, in dollar, yen, gold, or whatever else. All that matter is the *relative price* between goods. Money plays no role in general equilibrium theory, at least in the basic Arrow-Debreu world. This property is known as the *neutrality of money.*

Objective of agents and equilibrium Agents are assumed to be rational and selfish, like an economics professor. Furthermore, agents are considered "small" compared to the whole economy and therefore we assume that they

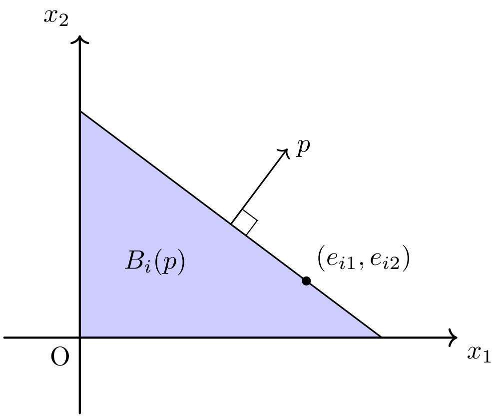

Figure 1.1: Budget set.

take price as given. Thus the objective of agents is to maximize utility subject to the budget constraint. Mathematically, agent i solves

$$\begin{array}{ll} \text{maximize} & u_i(x) \\ \text{subject to} & x \in B_i(p), \end{array}$$

which is a constrained maximization problem.

Now we can define an equilibrium of the economy.

Definition 1.2. A *competitive equilibrium* (also known as Walrasian equilibrium[5] or Arrow-Debreu equilibrium) $\{p, (x_i)\}$ consists of a price vector $p \in \mathbb{R}^L_+$ and an allocation $(x_i) \subset \mathbb{R}^L_+$ such that

(i) (Agent optimization) for each i, x_i solves the utility maximization problem, that is, $x_i \in B_i(p)$ and

$$x \in B_i(p) \implies u_i(x_i) \geq u_i(x),$$

(ii) (Market clearing) the allocation is feasible, that is,

$$\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i.$$

The idea of an equilibrium is as follows. A price p is quoted. Given the price, agents maximize utility and demand consumption bundles. If their demand is feasible, that is, for each good the total demand (*aggregate demand*) is less than or equal to the total endowment (*aggregate endowment*), then we say that the quoted price p and the resulting demand constitute an equilibrium. If the aggregate demand is not feasible, by definition there is at least one good in excess demand, and the price of that good is likely to rise. Thus an "equilibrium" is

[5] After Léon Walras (http://en.wikipedia.org/wiki/Leon_Walras), who proposed the concept. Arrow and Debreu (1954) provided the first rigorous proof of the existence of an equilibrium.

so called in the sense that there is no tendency for prices to move away from that point.

As soon as we define an equilibrium, a few fundamental questions arise. First, does an equilibrium always exist? The answer is yes, under reasonable assumptions. I will come back to this issue in Chapter 6. Second, is an equilibrium unique? The answer is no, and in general the best we can prove is that there are only a finite number of equilibria. I will come back to this issue in Chapter 7. Third, how does an economy get to an equilibrium? This is a difficult question which I will not discuss further.

1.3 Examples

1.3.1 Two agent Cobb-Douglas economy

Consider an economy with two agents and two goods (a two-good, two-agent economy is often called an *Edgeworth box* economy). Agents 1 and 2 have utility functions

$$\begin{aligned} u_1(x_1, x_2) &= \frac{2}{3}\log x_1 + \frac{1}{3}\log x_2, \\ u_2(x_1, x_2) &= \frac{1}{3}\log x_1 + \frac{2}{3}\log x_2, \end{aligned}$$

respectively. Thus agent 1 values good 1 more and vice versa. The initial endowment is $e_1 = e_2 = (3, 3)$, so both agents start with 3 units of each good. Let's compute the equilibrium.

Both agents have a Cobb-Douglas utility function. Since it shows up so often, it is useful to derive a general formula. Consider the maximization problem

$$\begin{aligned} &\text{maximize} && \alpha \log x_1 + (1-\alpha)\log x_2 \\ &\text{subject to} && p_1 x_1 + p_2 x_2 \le w, \end{aligned}$$

where x_1, x_2 are consumption of goods 1 and 2, p_1, p_2 are prices of goods 1 and 2, $\alpha > 0$ is a parameter ($\alpha = 2/3$ for agent 1 and $\alpha = 1/3$ for agent 2), and w is wealth. The Lagrangian of this problem is

$$L(x_1, x_2, \lambda) = \alpha \log x_1 + (1-\alpha)\log x_2 + \lambda(w - p_1 x_1 - p_2 x_2).$$

The first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{\alpha}{x_1} - \lambda p_1 = 0 \iff x_1 = \frac{\alpha}{\lambda p_1}, \\ \frac{\partial L}{\partial x_2} &= \frac{1-\alpha}{x_2} - \lambda p_2 = 0 \iff x_2 = \frac{1-\alpha}{\lambda p_2}. \end{aligned}$$

By the budget constraint (complementary slackness), we have

$$w = p_1 x_1 + p_2 x_2 = \frac{\alpha}{\lambda} + \frac{1-\alpha}{\lambda} \iff \lambda = \frac{1}{w}.$$

Therefore the demand is

$$(x_1, x_2) = \left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right).$$

Now we put all pieces together. Substituting $\alpha = 2/3$ and $w = 3p_1 + 3p_2$, the demand of agent 1 is

$$(x_{11}, x_{12}) = \left(\frac{2p_1 + 2p_2}{p_1}, \frac{p_1 + p_2}{p_2}\right).$$

Substituting $\alpha = 1/3$ and $w = 3p_1 + 3p_2$, the demand of agent 2 is

$$(x_{21}, x_{22}) = \left(\frac{p_1 + p_2}{p_1}, \frac{2p_1 + 2p_2}{p_2}\right).$$

By market clearing for good 1 with equality (see Theorem 3.4 for a rigorous discussion), we get

$$\begin{aligned} x_{11} + x_{21} = e_{11} + e_{21} &\iff \frac{2p_1 + 2p_2}{p_1} + \frac{p_1 + p_2}{p_1} = 3 + 3 \\ &\iff 3(p_1 + p_2) = 6p_1 \iff p_1 = p_2. \end{aligned}$$

(We obtain the same relation if we invoke market clearing for good 2.) Substituting the price into agents' demand, the equilibrium is

$$\{(p_1, p_2), ((x_{11}, x_{12}), (x_{21}, x_{22}))\} = \{(t, t), ((4, 2), (2, 4))\},$$

where $t > 0$ is arbitrary. The outcome is quite natural. Since agent 1 values good 1 more and agents are equally rich, agent 1 ends up buying good 1 and selling good 2. The reason why the equilibrium price *level* is indeterminate is due to the neutrality of money. However, the *relative price* of the two goods is determinate. In this case, $p_2/p_1 = 1$.

1.3.2 Interest rate

Recall that goods must be distinguished not just by physical properties but also by time, location, and states of the world. The next example shows how we can apply general equilibrium theory to derive implications for the interest rate.

Consider an economy consisting of two periods denoted by $t = 0, 1$. In each period, there is a single perishable good (e.g., raw fish). For simplicity, assume that there is only one agent (or many identical agents) with endowment $e = (e_0, e_1)$ and utility function

$$u(x_0, x_1) = \log x_0 + \beta \log x_1,$$

where $\beta > 0$ is called the discount factor. When β is large, the agent puts more weight on the future, so he is patient. The agent is impatient when β is small.

Let the prices of the goods be $p_0 = 1$ and $p_1 = p$. In this case, all markets open at $t = 0$. Here p_0 is the spot price of the good at $t = 0$ and p_1 is the price of one future contract to have one unit of good delivered at $t = 1$. (Imagine that the goods are wheat, corn, or coffee beans, and the agents are trading in the spot market as well as in commodity futures.)

In order to apply the Cobb-Douglas formula, the weights on the goods must sum to 1. Therefore instead of the utility function given above, we can use

$$v(x_0, x_1) = \frac{1}{1 + \beta} \log x_0 + \frac{\beta}{1 + \beta} \log x_1,$$

which is a monotonic transformation of u (by dividing it by $1+\beta$). Using the Cobb-Douglas formula for $\alpha = \frac{1}{1+\beta}$, the demand is

$$(x_0, x_1) = \left(\frac{1}{1+\beta}\frac{e_0+pe_1}{1}, \frac{\beta}{1+\beta}\frac{e_0+pe_1}{p}\right).$$

The supply is (e_0, e_1). Setting the demand equal to supply, the equilibrium price is

$$x_0 = e_0 \iff \frac{1}{1+\beta}\frac{e_0+pe_1}{1} = e_0 \iff p = \beta\frac{e_0}{e_1}.$$

As an application, let us compute the interest rate in this economy. If the net interest rate is r, when you save \$1 it grows to \$$(1+r)$ at the end of the period. Here we are interested in the real interest rate: how much good you can get at $t=1$ by forgoing one unit of good at $t=0$. By the definition of the real interest rate, in order to get one unit of good at $t=1$, you must save $\frac{1}{1+r}$ unit of good at $t=0$. Alternatively, in order to get one unit of good at $t=1$, you can just buy one unit of futures contract, whose price is p. Therefore it must be the case that

$$\frac{1}{1+r} = p \iff 1+r = \frac{1}{p} = \frac{1}{\beta}\frac{e_1}{e_0}.$$

This formula tells us that the interest rate is high when people are impatient (low β) or the economy grows fast (high e_1/e_0). Although this model is very simple, we can already see how general equilibrium theory serves as the foundation of applied fields.

Chapter 2

Convex analysis and convex programming

2.1 Convex sets

A set $C \subset \mathbb{R}^N$ is said to be *convex* if the line segment generated by any two points in C is entirely contained in C. Formally, C is convex if $x, y \in C$ implies $(1-\alpha)x + \alpha y \in C$ for all $\alpha \in [0,1]$ (Figure 2.1). So a circle, triangle, and square are convex but a star-shape is not (Figure 2.2). One of my favorite mathematical jokes is that the Chinese character for "convex" is not convex (Figure 2.3).

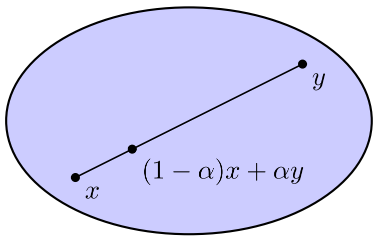

Figure 2.1: Definition of a convex set.

2.2 Separation of convex sets

You should know from high school that the equation of a line in $\mathbb{R}^2$ is

$$a_1 x_1 + a_2 x_2 = c$$

for some real numbers a_1, a_2, c, and that the equation of a plane in $\mathbb{R}^3$ is

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = c.$$

Letting $a = (a_1, \dots, a_N)$ and $x = (x_1, \dots, x_N)$ be vectors in $\mathbb{R}^N$, the equation $\langle a, x \rangle = c$ is a line if $N = 2$ and a plane if $N = 3$, where

$$\langle a, x \rangle = a_1 x_1 + \dots + a_N x_N$$

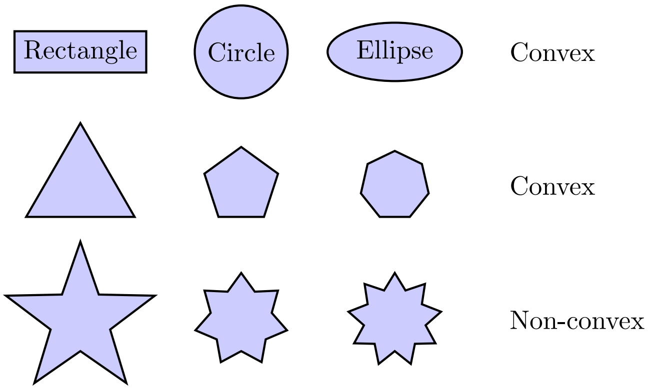

Figure 2.2: Examples of convex and non-convex sets.

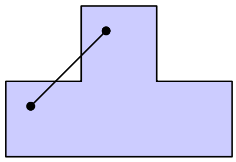

Figure 2.3: Chinese character for "convex" is not convex.

is the inner product of the vectors a and x.[1] In general, we say that the set

$$\left\{x \in \mathbb{R}^N \,\middle|\, \langle a, x\rangle = c\right\}$$

is a *hyperplane* if $a \neq 0$. The vector a is orthogonal to this hyperplane (is a *normal vector*). To see this, let x_0 be a point in the hyperplane. Since $\langle a, x_0\rangle = c$, by subtraction and linearity of inner product we get $\langle a, x - x_0\rangle = 0$. This means that the vector a is orthogonal to the vector $x - x_0$, which can point to any direction in the plane by moving x. So it makes sense to say that a is orthogonal to the hyperplane $\langle a, x\rangle = c$. The sets

$$\begin{aligned} H^+ &= \left\{x \in \mathbb{R}^N \,\middle|\, \langle a, x\rangle \geq c\right\}, \\ H^- &= \left\{x \in \mathbb{R}^N \,\middle|\, \langle a, x\rangle \leq c\right\} \end{aligned}$$

are called *half spaces*, since H^+ (H^-) is the portion of $\mathbb{R}^N$ separated by the hyperplane $\langle a, x\rangle = c$ towards the direction of a ($-a$). Hyperplanes and half spaces are convex sets (exercise).

Let A, B be two sets. We say that the hyperplane $\langle a, x\rangle = c$ *separates* A, B if $A \subset H^-$ and $B \subset H^+$ (Figure 2.4), that is,

$$\begin{aligned} x \in A &\implies \langle a, x\rangle \leq c, \\ x \in B &\implies \langle a, x\rangle \geq c. \end{aligned}$$

(The inequalities may be reversed.)

[1] The inner product is sometimes called the *vector product* or the *dot product*. Common notations for the inner product are $\langle a, x\rangle$, (a, x), $a \cdot x$, etc.

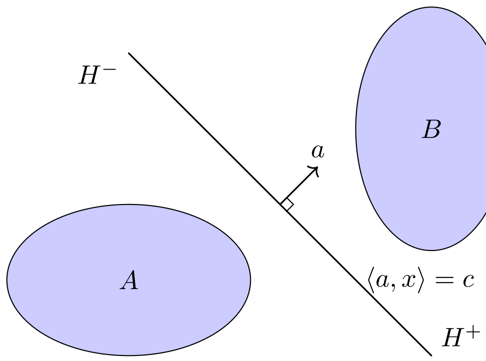

Figure 2.4: Separation of convex sets.

Clearly A, B can be separated if and only if

$$\sup_{x \in A} \langle a, x \rangle \leq \inf_{x \in B} \langle a, x \rangle ,$$

since we can take c between these two numbers. We say that A, B can be *strictly separated* if the inequality is strict, so

$$\sup_{x \in A} \langle a, x \rangle < \inf_{x \in B} \langle a, x \rangle .$$

The remarkable property of convex sets is the following separation property.

Theorem 2.1 (Separating Hyperplane Theorem)**.** *Let $C, D \subset \mathbb{R}^N$ be nonempty and convex. If $C \cap D = \emptyset$, then there exists a hyperplane that separates C, D. If C, D are closed and one of them is compact, then they can be strictly separated.*

We need the following lemma to prove Theorem 2.1.

Lemma 2.2. *Let C be nonempty and convex. Then any $x \in \mathbb{R}^N$ has a unique closest point $P_C(x) \in \operatorname{cl} C$, called the* projection of x on $\operatorname{cl} C$*. Furthermore, for any $z \in C$ we have*

$$\langle x - P_C(x), z - P_C(x) \rangle \leq 0.$$

Proof. Let $\delta = \inf \{\|x - y\| \mid y \in C\} \geq 0$ be the distance from x to C (Figure 2.5).

Take a sequence $\{y_k\} \subset C$ such that $\|x - y_k\| \to \delta$. Then by simple algebra we get

$$\|y_k - y_l\|^2 = 2 \|x - y_k\|^2 + 2 \|x - y_l\|^2 - 4 \left\| x - \frac{1}{2}(y_k + y_l) \right\|^2 . \tag{2.1}$$

Since C is convex, we have $\frac{1}{2}(y_k + y_l) \in C$, so by the definition of δ we get

$$\|y_k - y_l\|^2 \leq 2 \|x - y_k\|^2 + 2 \|x - y_l\|^2 - 4\delta^2 \to 2\delta^2 + 2\delta^2 - 4\delta^2 = 0$$

as $k, l \to \infty$. Since $\{y_k\} \subset C$ is Cauchy, it converges to some point $y \in \operatorname{cl} C$. Then

$$\|x - y\| \leq \|x - y_k\| + \|y_k - y\| \to \delta + 0 = \delta,$$

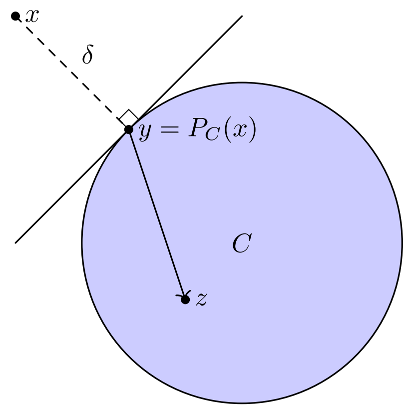

Figure 2.5: Projection on a convex set.

so y is the closest point to x in $\operatorname{cl} C$. If y_1, y_2 are two closest points, then by the same argument we get

$$\|y_1 - y_2\|^2 \le 2\|x - y_1\|^2 + 2\|x - y_2\|^2 - 4\delta^2 \le 0,$$

so $y_1 = y_2$. Thus $y = P_C(x)$ is unique.

Finally, let $z \in C$ be any point. Take $\{y_k\} \subset C$ such that $y_k \to y = P_C(x)$. Since C is convex, for any $0 < \alpha \le 1$ we have $(1-\alpha)y_k + \alpha z \in C$. Therefore

$$\delta^2 = \|x - y\|^2 \le \|x - (1-\alpha)y_k - \alpha z\|^2 .$$

Letting $k \to \infty$ we get $\|x - y\|^2 \le \|x - y - \alpha(z - y)\|^2$. Expanding both sides, dividing by $\alpha > 0$, and letting $\alpha \to 0$, we get $\langle x - y, z - y\rangle \le 0$, which is the desired inequality. $\square$

The following proposition shows that a point that is not an interior point of a convex C can be separated from C.

Proposition 2.3. *Let $C \subset \mathbb{R}^N$ be nonempty and convex and $\bar{x} \notin \operatorname{int} C$. Then there exists a hyperplane $\langle a, x\rangle = c$ that separates $\bar{x}$ and C, i.e.,*

$$\langle a, \bar{x}\rangle \ge c \ge \langle a, z\rangle$$

for any $z \in C$. If $\bar{x} \notin \operatorname{cl} C$, then the above inequalities can be made strict.

Proof. Suppose that $\bar{x} \notin \operatorname{cl} C$. Let $y = P_C(\bar{x})$ be the projection of $\bar{x}$ on $\operatorname{cl} C$. Then $\bar{x} \ne y$. Let $a = \bar{x} - y \ne 0$ and $c = \langle a, y\rangle + \frac{1}{2}\|a\|^2$. Then for any $z \in C$ we have

$$\langle \bar{x} - y, z - y\rangle \le 0 \implies \langle a, z\rangle \le \langle a, y\rangle < \langle a, y\rangle + \frac{1}{2}\|a\|^2 = c,$$
$$\langle a, \bar{x}\rangle - c = \langle \bar{x} - y, \bar{x} - y\rangle - \frac{1}{2}\|a\|^2 = \frac{1}{2}\|a\|^2 > 0 \iff \langle a, \bar{x}\rangle > c.$$

Therefore the hyperplane $\langle a, x\rangle = c$ strictly separates $\bar{x}$ and C.

If $\bar{x} \in \operatorname{cl} C$, since $\bar{x} \notin \operatorname{int} C$ we can take a sequence $\{x_k\}$ such that $x_k \notin \operatorname{cl} C$ and $x_k \to \bar{x}$. Then we can find a vector $a_k \neq 0$ and a number $c_k \in \mathbb{R}$ such that

$$\langle a_k, x_k \rangle \geq c_k \geq \langle a_k, z \rangle$$

for all $z \in C$. By dividing both sides by $\|a_k\| \neq 0$, without loss of generality we may assume $\|a_k\| = 1$. Since $x_k \to \bar{x}$, the sequence $\{c_k\}$ is bounded. Therefore we can find a convergent subsequence $(a_{k_l}, c_{k_l}) \to (a, c)$. Letting $l \to \infty$, we get

$$\langle a, \bar{x} \rangle \geq c \geq \langle a, z \rangle$$

for any $z \in C$. □

Proof of Theorem 2.1. Let $E = C - D = \{x - y \mid x \in C, y \in D\}$. Since C, D are nonempty and convex, so is E. Since $C \cap D = \emptyset$, we have $0 \notin E$. In particular, $0 \notin \operatorname{int} E$. By Proposition 2.3, there exists $a \neq 0$ such that $\langle a, 0 \rangle = 0 \geq \langle a, z \rangle$ for all $z \in E$. By the definition of E, we have

$$\langle a, x - y \rangle \leq 0 \iff \langle a, x \rangle \leq \langle a, y \rangle$$

for any $x \in C$ and $y \in D$. Letting $\sup_{x \in C} \langle a, x \rangle \leq c \leq \inf_{y \in D} \langle a, y \rangle$, it follows that the hyperplane $\langle a, x \rangle = c$ separates C and D.

Suppose that C is closed and D is compact. Let us show that $E = C - D$ is closed. For this purpose, suppose that $\{z_k\} \subset E$ and $z_k \to z$. Then we can take $\{x_k\} \subset C$, $\{y_k\} \subset D$ such that $z_k = x_k - y_k$. Since D is compact, there is a subsequence such that $y_{k_l} \to y \in D$. Then $x_{k_l} = y_{k_l} + z_{k_l} \to y + z$, but since C is closed, $x = y + z \in C$. Therefore $z = x - y \in E$, so E is closed.

Since $E = C - D$ is closed and $0 \notin E$, by Proposition 2.3 there exists $a \neq 0$ such that $\langle a, 0 \rangle = 0 > \langle a, z \rangle$ for all $z \in E$. The rest of the proof is similar. □

2.3 Convex programming

Let $f : \mathbb{R}^N \to [-\infty, \infty]$ be a function. The set

$$\operatorname{epi} f := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R} \,\middle|\, f(x) \leq y \right\}$$

is called the *epigraph* of f, for the obvious reason that $\operatorname{epi} f$ is the set of points that lie on or above the graph of f. f is said to be *convex* if $\operatorname{epi} f$ is convex. f is convex if and only if for any $x_1, x_2 \in \mathbb{R}^N$ and $\alpha \in [0, 1]$ we have (exercise)

$$f((1 - \alpha)x_1 + \alpha x_2) \leq (1 - \alpha) f(x_1) + \alpha f(x_2).$$

This inequality is often used as the definition of a convex function. When this inequality is strict whenever $x_1 \neq x_2$ and $0 < \alpha < 1$, then f is said to be *strictly convex.*

Another useful but weaker concept is quasi-convexity. The set

$$L_f(y) = \left\{ x \in \mathbb{R}^N \,\middle|\, f(x) \leq y \right\}$$

is called the *lower contour set* of f at level y. f is said to be *quasi-convex* if all lower contour sets are convex. f is quasi-convex if and only if for any $x_1, x_2 \in \mathbb{R}^N$ and $\alpha \in [0, 1]$ we have (exercise)

$$f((1 - \alpha)x_1 + \alpha x_2) \leq \max \{f(x_1), f(x_2)\}.$$

Again if the inequality is strict whenever $x_1 \neq x_2$ and $0 < \alpha < 1$, then f is said to be *strictly quasi-convex*.

f is said to be *concave* if $-f$ is convex, that is, f is a convex function flipped up side down. The definition for strict concavity or quasi-concavity is similar.

In economics, we often encounter constrained optimization problems of the form

$$\begin{aligned} &\text{maximize} && f(x) \\ &\text{subject to} && g_k(x) \geq 0, && (k = 1, \dots, K) \end{aligned} \tag{2.2}$$

where f is the *objective function* and $g_k(x) \geq 0$ is a *constraint*. For instance, the utility maximization problem

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && x \in B_i(p) \end{aligned}$$

fits into this form by letting $f(x) = u_i(x)$, $g_l(x) = x_l$ $(l = 1, \dots, L)$, $g_{L+1}(x) = p \cdot (e_i - x)$, and setting $K = L + 1$.

There is a procedure to solve such constrained maximization problems, known as the *Karush-Kuhn-Tucker* (KKT) theorem. Remember from calculus that the vector of partial derivatives,

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right]',$$

is called the *gradient* of f.

Theorem 2.4 (Karush-Kuhn-Tucker for concave functions)**.** *Let f, g_k's be concave and differentiable.*

(i) *If $\bar{x}$ is a solution to the optimization problem (2.2) and there exists a point x_0 such that $g_k(x_0) > 0$ for all k, then there exists $\lambda = (\lambda_1, \dots, \lambda_K) \in \mathbb{R}^K_+$ such that*

$$\nabla f(\bar{x}) + \sum_{k=1}^{K} \lambda_k \nabla g_k(\bar{x}) = 0, \tag{2.3a}$$

$$(\forall k)\ \lambda_k \geq 0,\ \ g_k(\bar{x}) \geq 0,\ \ \lambda_k g_k(\bar{x}) = 0. \tag{2.3b}$$

(ii) *Conversely, if $\bar{x}$ and λ satisfy (2.3), then $\bar{x}$ is a solution to (2.2).*

The condition that there exists a point x_0 such that $g_k(x_0) > 0$ for all k is called the *Slater constraint qualification*. The vector $\lambda \in \mathbb{R}^K_+$ is called the *Lagrange multiplier*. The conditions (2.3a) and (2.3b) are called the *first-order condition* and the *complementary slackness condition*, and jointly the Karush-Kuhn-Tucker (KKT) conditions. The KKT theorem says that if the constraint qualification holds, then a solution satisfies the KKT conditions. Conversely, if a point satisfies the KKT conditions, then it is a solution (regardless of the constraint qualification). The proof of the KKT theorem is not so simple, so refer to my lecture notes for Math Camp (Econ 205)[2] if you are interested.

[2] https://alexisakira.github.io/teaching/mathcamp

There is also a version of the KKT theorem when the functions f and g_k's are only quasi-concave. When f, g_k's are only differentiable, the KKT conditions are necessary for optimality under some regularity condition, but not sufficient.

The KKT theorem implies that we can solve the optimization problem (2.2) by following the steps below.

Step 1. Verify the Slater constraint qualification.

Step 2. Define the Lagrangian by

$$L(x, \lambda) = f(x) + \sum_{k=1}^{K} \lambda_k g_k(x).$$

Step 3. Derive the first-order condition (2.3a) (which is $\partial L/\partial x_n = 0$ for all n) and the complementary slackness condition (2.3b).

Step 4. Solve these $N+K$ equations in $N+K$ unknowns ($x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}^K$). If there is a solution, $\bar{x}$ is a solution to (2.2). Otherwise, there is no solution.

In economic applications, the constraint functions g_k are often linear (more precisely, affine), meaning that $g_k(x) = \langle a_k, x\rangle - b_k$ for some vector a_k and number b_k. The utility maximization problem is such an example (exercise). In that case, we can dispose of the constraint qualification, as the following theorem shows. (Proof in Econ 172B lecture notes.)

Theorem 2.5 (Karush-Kuhn-Tucker with linear constraints)**.** *Let f be concave and g_k's linear. Then $\bar{x}$ is a solution to the optimization problem* (2.2) *if and only if there exists $\lambda = (\lambda_1, \dots, \lambda_K) \in \mathbb{R}^K_+$ such that the KKT conditions* (2.3) *hold.*

Now consider the utility maximization problem. If we want to apply the KKT theorem literally, we need to set up $L+1$ Lagrange multipliers—one for the budget constraint and L for the nonnegativity constraints.[3] Oftentimes, the nonnegativity constraints are not binding (they are strict inequalities), in which case we know that the Lagrange multipliers must be zero by complementary slackness. A well-known case is the following Inada condition.[4]

Inada condition. We say that u satisfies the Inada condition for good l if

$$\lim_{x_l \to +0} \frac{\partial u_i(x)}{\partial x_l} = \infty.$$

If the Inada condition holds, a consumer will never want to consume nothing of one good, because he can increase his utility by consuming a tiny bit of that good and reducing other goods to satisfy the budget constraint. Here is the formal statement.

Proposition 2.6. *Suppose that the utility function u_i is continuous on $\mathbb{R}^L_+$, differentiable on $\mathbb{R}^L_{++}$, and the partial derivatives $\partial u_i/\partial x_l$ are bounded from below. Suppose that $p \gg 0$ and x_i is the solution to the utility maximization problem. If the Inada condition is satisfied for good l, then $x_{il} > 0$.*

[3] By the way, the letter "L" has many roles—the number of goods, the lower contour set, and the Lagrangian—but the meaning should be clear from the context.

[4] http://en.wikipedia.org/wiki/Inada_conditions lists more conditions.

Proof. Take a vector $d \in \mathbb{R}^L$ such that $d_l > 0$ whenever $x_{il} = 0$ and $p \cdot (x_i + td) \leq w$ for sufficiently small $t > 0$. (There are many such examples: if $x_i = 0$, just take $d = (1, \dots, 1)'$; if $x_{il'} > 0$ for some l', take $d_{l'} = -1$ and $d_l = \epsilon > 0$ for all $l \neq l'$ for sufficiently small $\epsilon > 0$.)

Suppose to the contrary that the Inada condition holds for good l but $x_{il} = 0$. By construction, we have $d_l > 0$. By the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$\begin{aligned} & u_i(x_i + td) - u_i(x_i) = \nabla u_i(x_i + \theta td) \cdot (td) \\ \iff & \frac{u_i(x_i + td) - u_i(x_i)}{t} = \nabla u_i(x_i + \theta td) \cdot d. \end{aligned}$$

Since the partial derivatives are bounded below (say by $-M < 0$), it follows that

$$\frac{u_i(x_i + td) - u_i(x_i)}{t} \geq -M \sum_{l' \neq l} |d_l| + \frac{\partial u_i(x_i + \theta td)}{\partial x_l} d_l \to \infty$$

as $t \to 0$ by the Inada condition and $d_l > 0$. Therefore for sufficiently small $t > 0$, we have $u_i(x_i + td) > u_i(x_i)$. Since by construction $p \cdot (x_i + td) \leq w$, the bundle $x_i + td$ is affordable and gives higher utility than x_i, which is a contradiction. Therefore $x_{il} > 0$. $\square$

Therefore the nonnegativity constraints do not bind, and we obtain the following corollary.

Corollary 2.7. *Assume that the utility function u_i is concave, differentiable, and satisfies the Inada condition for all goods. If x_i is the solution to the utility maximization problem, then there exists $\lambda \geq 0$ such that*

$$\frac{\partial u_i(x_i)}{\partial x_l} = \lambda p_l$$

for all l.

Proof. By the Inada condition, the nonnegativity constraints do not bind. Therefore the Lagrangian is

$$L(x, \lambda) = u_i(x) + \lambda(w - p \cdot x),$$

where $w = p \cdot e_i$. The first-order condition with respect to x_l is $\frac{\partial u_i(x_i)}{\partial x_l} = \lambda p_l$. $\square$

Example 2.1 (Cobb-Douglas utility)**.** Consider the Cobb-Douglas utility function

$$u(x) = \sum_{l=1}^{L} \alpha_l \log x_l,$$

where $\alpha_l > 0$ and $\sum_{l=1}^{L} \alpha_l = 1$. Let us compute the demand. $u(x)$ is concave since it is the weighted sum of concave functions $\log x_l$. Furthermore, $\partial u / \partial x_l = \alpha_l / x_l \to \infty$ as $x_l \to 0$, so it satisfies the Inada condition.

Let

$$L(x, \lambda) = \sum_{l=1}^{L} \alpha_l \log x_l + \lambda(w - p \cdot x)$$

be the Lagrangian. The first-order condition with respect to x_l is

$$0 = \frac{\partial L}{\partial x_l} = \frac{\alpha_l}{x_l} - \lambda p_l \iff x_l = \frac{\alpha_l}{\lambda p_l}.$$

(Note that if $\lambda = 0$, then $\alpha_l / x_l = 0$, which is a contradiction since $\alpha_l > 0$.) The complementary slackness condition is

$$0 = \lambda(w - p \cdot x) = \lambda w - 1 \iff \lambda = \frac{1}{w}.$$

Therefore

$$x_l = \frac{\alpha_l}{\lambda p_l} = \frac{\alpha_l w}{p_l},$$

which is the usual Cobb-Douglas formula.

Chapter 3

Walras law

3.1 How to solve for equilibrium

In the examples in Section 1.3, we were able to compute the equilibrium by invoking market clearing (with equality) for only one good. Does this generalize? The answer is yes, most of the time.

To answer this question, we need some definitions. A utility function $u : \mathbb{R}_+^L \to \mathbb{R}$ is said to be

strongly monotonic if $y > x$ implies $u(y) > u(x)$,

weakly monotonic if $y \geq x$ implies $u(y) \geq u(x)$ and $y \gg x$ implies $u(y) > u(x)$, and

locally nonsatiated if for any x and any $\epsilon > 0$, there exists y such that $\|y - x\| < \epsilon$ and $u(y) > u(x)$.

Strong monotonicity says that you are better off if you consume more of any good. Real people are not like that (think of eating another hamburger after having eaten 10 hamburgers). Weak monotonicity says that you are better off if you consume strictly more of all goods. This is a good assumption if goods are "good" (i.e., no garbage). Local nonsatiation says that for any consumption bundle, you can find an arbitrarily close bundle with which you are better off. This assumption holds even if there is garbage. It is not difficult to prove that strong monotonicity implies weak monotonicity, which implies local nonsatiation.

The following proposition shows that an agent with locally nonsatiated utility function spends all his income.

Proposition 3.1. *Let $u : \mathbb{R}_+^L \to \mathbb{R}$ be a locally nonsatiated utility function and $x(p, w)$ be a solution to the utility maximization problem*

$$\begin{aligned} &\text{maximize} && u(x) \\ &\text{subject to} && p \cdot x \leq w,\ x \in \mathbb{R}_+^L. \end{aligned}$$

Then $p \cdot x(p, w) = w$.

Proof. Let $x = x(p, w)$. Assume that $p \cdot x < w$ and let $\epsilon = \frac{w - p \cdot x}{\|p\|} > 0$. By local nonsatiation, there exists x' such that $\|x' - x\| < \epsilon$ and $u(x') > u(x)$. By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} p \cdot x' &= p \cdot x + p \cdot (x' - x) \leq p \cdot x + \|p\| \, \|x' - x\| \\ &< p \cdot x + \|p\| \, \epsilon = w, \end{aligned}$$

so x' is affordable. But this contradicts the optimality of x in the budget set. Therefore $p \cdot x = w$. □

By Proposition 3.1, we immediately obtain the following corollary.

Corollary 3.2 (Walras law). *Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy with locally nonsatiated utilities. Let $x_i(p, p \cdot e_i)$ be the demand of agent i with wealth $w_i = p \cdot e_i$ and*

$$z(p) = \sum_{i=1}^{I} (x_i(p, p \cdot e_i) - e_i)$$

be the aggregate excess demand. Then $p \cdot z(p) = 0$.

Proof. Let $w_i = p \cdot e_i$. By Proposition 3.1, we have $p \cdot x_i(p, w_i) = w_i \iff p \cdot (x_i(p, p \cdot e_i) - e_i) = 0$. Summing this equation over i yields $p \cdot z(p) = 0$. □

The following proposition shows that in equilibrium, prices are nonnegative and a good in excess supply must be free.

Proposition 3.3. *Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with locally nonsatiated utilities and $\{p, (x_i)\}$ be a Walrasian equilibrium. Then $p_l \geq 0$ and*

$$p_l \sum_{i=1}^{I} (x_{il} - e_{il}) = 0$$

for all l. In particular, $p_l = 0$ if $\sum_{i=1}^{I} (x_{il} - e_{il}) < 0$.

Proof. Suppose $p_l < 0$ for some l. By buying $\epsilon > 0$ of good l and then throwing it away, an agent pays $p_l \epsilon < 0$ so receives $-p_l \epsilon > 0$ units of account. Since agents have locally nonsatiated utility functions, the agent can spend this extra income on goods and increase utility, which contradicts utility maximization. Therefore $p_l \geq 0$ for all l. Since $\{p, (x_i)\}$ is a Walrasian equilibrium, we have $\sum_{i=1}^{I} (x_{il} - e_{il}) \leq 0$ for all l, so $p_l \sum_{i=1}^{I} (x_{il} - e_{il}) \leq 0$. If $p_l \sum_{i=1}^{I} (x_{il} - e_{il}) < 0$ for some l, adding across l we obtain

$$0 > \sum_{l=1}^{L} p_l \sum_{i=1}^{I} (x_{il} - e_{il}) = \sum_{i=1}^{I} \sum_{l=1}^{L} p_l (x_{il} - e_{il}) = \sum_{i=1}^{I} p \cdot (x_i - e_i) = 0$$

by Walras law, which is a contradiction. □

The following theorem shows that we can solve for the equilibrium by clearing all but one markets because the other market automatically clears.

Theorem 3.4. *Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with locally nonsatiated utilities. Assume that at least one agent has a strongly monotonic preference. Let x_i be the demand of agent i (solution of utility maximization problem) given price p. Then $\{p, (x_i)\}$ is an equilibrium if and only if*

$$\sum_{i=1}^{I} (x_{il} - e_{il}) = 0$$

for $l = 1, \dots, L-1$.

Proof. If $\{p, (x_i)\}$ is an equilibrium, by Proposition 3.3, we have $p_l \sum_{i=1}^{I} (x_{il} - e_{il}) = 0$ for all l. If $p_l = 0$, the agent with strongly monotonic preferences will demand an infinite amount of good l, which violates market clearing. Therefore $p_l > 0$ and $\sum_{i=1}^{I} (x_{il} - e_{il}) = 0$ for all l.

Conversely, suppose $\sum_{i=1}^{I} (x_{il} - e_{il}) = 0$ for $l = 1, \dots, L-1$. By the Walras law, we obtain

$$0 = p \cdot \sum_{i=1}^{I} (x_i - e_i) = \sum_{l=1}^{L} p_l \sum_{i=1}^{I} (x_{il} - e_{il}) = p_L \sum_{i=1}^{I} (x_{iL} - e_{iL}).$$

If $p_L = 0$, the agent with strongly monotonic preferences will demand an infinite amount of good L, which contradicts the assumption that a solution exists. Therefore $p_L > 0$ and $\sum_{i=1}^{I} (x_{iL} - e_{iL}) = 0$, so $\{p, (x_i)\}$ is an equilibrium. $\square$

In most applications, agents have strongly monotonic preferences. Therefore if an equilibrium exists, prices must be positive, and all markets must clear with no excess supply. But the previous corollary says that if all but one markets clear, so does the other one. Therefore we can do as follows to find the equilibrium.

Step 1. For each agent i, solve the utility maximization problem

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \leq w. \end{aligned}$$

Let $x_i(p, w)$ be the solution.

Step 2. Set $p_1 = 1$, $w = p \cdot e_i$, and solve the system of $L-1$ equations in $L-1$ unknowns $(p_2, \dots, p_L)$,

$$\sum_{i=1}^{I} (x_{il}(p, p \cdot e_i) - e_{il}) = 0, \ \ (l = 1, \dots, L-1).$$

Of course, by the neutrality of money the price level is indeterminate. Therefore without loss of generality we can fix one price, say $p_1 = 1$.

3.2 Production economy

So far, we have studied an *exchange economy*, where agents are endowed with goods and exchange them through markets. In the real world there is production, which we model in this section.

At the most abstract level, a technology (production possibility) is just modeled as a subset Y of the commodity space $\mathbb{R}^L$. Let $y = (y_1, \dots, y_L) \in Y$. By convention, good l is an input if $y_l < 0$ and is an output if $y_l > 0$. For instance, if $L = 2$ and $y = (-2, 1) \in Y$, it means that we can produce one unit of good 2 from two units of good 1. Similarly if $L = 3$ and $y = (-2, -3, 1) \in Y$, we can produce one unit of good 3 from two units of good 1 and three units of good 2.

Firms are indexed by $j = J \in \{1, \dots, J\}$. Firm j is characterized by a production possibility set $Y_j \subset \mathbb{R}^L$. If firm j chooses a production plan $y \in Y_j$ when the price vector is p, its profit is

$$\text{revenue} - \text{cost} = \sum_{l:y_l>0} p_l y_l - \sum_{l:y_l<0} p_l(-y_l) = \sum_{l=1}^{L} p_l y_l = p \cdot y,$$

simply the inner product of the price and the input-output vector.

The firm's profit must go somewhere. In an Arrow-Debreu model, we assume that consumers are shareholders and the profit goes to the shareholders according to the ownership share. Let $\theta_{ij} \geq 0$ be the ownership share of agent i in firm j. By definition, $\sum_{i=1}^{I} \theta_{ij} = 1$. We are led to the definition of a production economy as follows.

Definition 3.5. An *Arrow-Debreu economy with production*

$$\mathcal{E} = \{I, J, (e_i), (u_i), (Y_j), (\theta_{ij})\}$$

consists of the set of agents $I = \{1, \dots, I\}$, their endowments $(e_i) \subset \mathbb{R}_+^L$ and utility functions $u_i : \mathbb{R}_+^L \to \mathbb{R}$, the set of firms $J = \{1, \dots, J\}$, their production possibility sets $Y_j \subset \mathbb{R}^L$, and the ownership share of agents (θ_{ij}), where $\theta_{ij} \geq 0$ and $\sum_{i=1}^{I} \theta_{ij} = 1$.

If firm j chooses a production plan $y_j \in Y_j$, its profit is $p \cdot y_j$, and therefore agent i receives the dividend $\theta_{ij} p \cdot y_j$. The budget constraint of agent i is then

$$\left\{ x \in \mathbb{R}_+^L \,\middle|\, p \cdot x \leq p \cdot e_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j \right\}.$$

Clearly agent i is better off the larger the profit is, because his budget will increase and therefore he will have more choices. Therefore all agents agree that firms should maximize their profits. Now we can define a competitive equilibrium.

Definition 3.6. A *competitive equilibrium* (Arrow-Debreu or Walrasian equilibrium)

$$\{p, (x_i), (y_j)\}$$

consists of a price vector $p \in \mathbb{R}_+^L$, an allocation $(x_i) \subset \mathbb{R}_+^L$, and a production plan $(y_j) \subset \mathbb{R}^L$ such that

(i) (Agent optimization) for each i, x_i solves the utility maximization problem

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \leq p \cdot e_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j, \ x \geq 0, \end{aligned}$$

(ii) (Profit maximization) for each j, y_j solves the profit maximization problem

$$\text{maximize } \ p \cdot y \ \text{ subject to } \ y \in Y_j,$$

(iii) (Market clearing) the allocation is feasible, that is,

$$\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i + \sum_{j=1}^{J} y_j.$$

In the course we will mostly work with exchange economies since the math is simpler, but occasionally treat production economies (especially in international trade and finance).

Chapter 4

Quasi-linear model

As we have seen in Section 3.1, finding the equilibrium is generally complicated because (i) we need to solve as many constrained optimization problems as the number of agents and then (ii) solve a system of nonlinear equations. In this chapter we study a special class of economies (*quasi-linear economies*) for which the computation of equilibrium is straightforward. Furthermore, the equilibrium has a certain welfare property.

4.1 Quasi-linear utilities

We say that a utility function u defined on $\mathbb{R} \times \mathbb{R}_+^L$ is *quasi-linear* if u has the form

$$u(x_0, x_1, \dots, x_L) = x_0 + \phi(x_1, \dots, x_L) \tag{4.1}$$

for some function $\phi : \mathbb{R}_+^L \to \mathbb{R}$. Here there is a special good 0, called the *numéraire* ("unit of account" in French), that can be consumed in positive or negative amounts. Note that the utility function is additively separable between good 0 and the rest, and the good 0 part is linear (hence the name quasi-linear). We can think of good 0 as money or gold. When agents have quasi-linear utilities, we always normalize the price of the numéraire good to be 1.

The following proposition shows that computing the demand for quasi-linear utility is relatively straightforward.

Proposition 4.1. *Let $\phi : \mathbb{R}_+^L \to \mathbb{R}$. And $p = (p_1, \dots, p_L)$ be the price vector of the non-numéraire goods. Then the solution to the utility maximization problem*

$$\begin{aligned} &\text{maximize} && x_0 + \phi(x) \\ &\text{subject to} && x_0 + p \cdot x \leq w \end{aligned}$$

is $(x_0, x) = (w - p \cdot x(p), x(p))$, where $x(p) \in \mathbb{R}_+^L$ solves

$$\max_x [\phi(x) - p \cdot x]. \tag{4.2}$$

In particular, the demand for the non-numéraire goods are independent of the wealth level w.

Proof. Since a quasi-linear utility function is locally nonsatiated (because it is strictly increasing in x_0), by Proposition 3.1 the budget constraint holds with equality, so $x_0 + p \cdot x = w$. Eliminating x_0 from the objective function, the utility maximization problem (UMP) is equivalent to

$$\max_x [\phi(x) + w - p \cdot x].$$

Since w is an additive constant, UMP reduces to (4.2). Letting $x = x(p)$ be its solution, it follows from the budget constraint that $x_0 = w - p \cdot x(p)$. □

Quasi-linear utility functions provide a formal justification of the partial equilibrium analysis taught in undergraduate intermediate microeconomics. To see this, suppose that $u : \mathbb{R} \times \mathbb{R}_+^L$ is the quasi-linear utility function (4.1), and suppose in addition that it is additively separable:

$$\phi(x) = \sum_{l=1}^{L} \phi_l(x_l),$$

where each $\phi_l : \mathbb{R}_+^L$ is assumed to be differentiable and satisfies $\phi_l' > 0$, $\phi_l'' < 0$, and the Inada condition $\phi_l'(0) = \infty$, $\phi_l'(\infty) = 0$. Since the objective function in (4.2) becomes additively separable as

$$\phi(x) - p \cdot x = \sum_{l=1}^{L} [\phi_l(x_l) - p_l x_l],$$

to maximize it, it suffices to maximize it term-by-term. The first-order condition for maximizing the l-th term is

$$\phi_l'(x_l) - p_l = 0 \iff x_l = (\phi_l')^{-1}(p_l).$$

Therefore the demand for good l depends only on its price, justifying partial equilibrium analysis.

Although quasi-linear utilities are useful for some applications, they are not realistic. As we can see from Proposition 4.1, as long as the price vector remains the same, a quasi-linear agent consumes constant amounts of non-numéraire goods, independent of his wealth. Hence the non-numéraire goods can be interpreted as basic necessities such as diapers. This is unrealistic. In reality, most people wear better clothes, eat better food, live in better places, and drive better cars as they become richer.

4.2 Equilibrium in quasi-linear economies

We now study the implication of quasi-linear utilities for the equilibrium. Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. We say that $\mathcal{E}$ is a *quasi-linear economy* if the commodity space is $\mathbb{R} \times \mathbb{R}_+^L$ and the utility functions are *quasi-linear*, so

$$u_i(x_0, \dots, x_L) = x_0 + \phi_i(x_1, \dots, x_L)$$

for some $\phi_i : \mathbb{R}_+^L \to \mathbb{R}$. The quasi-linear model is popular in applied work since the equilibrium is usually unique and can be easily computed. For notational simplicity, let $x = (x_1, \dots, x_L) \in \mathbb{R}^L_+$. Then the utility function is $u_i(x_0, x) = x_0 + \phi_i(x)$. With a slight abuse of notation, the endowment vector is then (e_{i0}, e_i). In a quasi-linear economy, agents are nonsatiated with the numéraire good, so $p_0 > 0$ in equilibrium. By the neutrality of money, without loss of generality we may assume $p_0 = 1$ by scaling the price level up or down. Let $p = (p_1, \dots, p_L)$ be the price vector of other goods. The following theorem not only tells us how to compute the equilibrium, but also is a mathematical formulation of Bentham's "greatest happiness principle":

> [...] fundamental axiom, it is the greatest happiness of the greatest number that is the measure of right and wrong.[1]

Theorem 4.2. *Suppose that ϕ_i's are continuous and concave on $\mathbb{R}^L_+$, differentiable on $\mathbb{R}^L_{++}$, and satisfy the Inada condition. Then $\left\{(1, p), (x_{i0}, x_i)_{i=1}^I\right\}$ is a Walrasian equilibrium if and only if $(x_i)_{i=1}^I$ solves the optimization problem*

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^I \phi_i(y_i) \\ &\text{subject to} && \sum_{i=1}^I (y_i - e_i) \le 0, \ (\forall i) y_i \ge 0, \end{aligned} \tag{4.3}$$

$p \in \mathbb{R}^L_+$ is the corresponding Lagrange multiplier, and $x_{i0} = e_{i0} + p \cdot (e_i - x_i)$. Furthermore, an equilibrium always exists, and is unique if all ϕ_i's are strictly concave.

Proof. Suppose that $\left\{(1, p), (x_{i0}, x_i)_{i=1}^I\right\}$ is a Walrasian equilibrium. Then for each i, (x_{i0}, x_i) solves the utility maximization problem

$$\begin{aligned} &\text{maximize} && y_{i0} + \phi_i(y_i) \\ &\text{subject to} && y_{i0} + p \cdot y_i \le e_{i0} + p \cdot e_i, \ y_i \ge 0. \end{aligned}$$

Since agents like the numéraire good, by Proposition 3.1 the budget constraint must hold with equality, so $y_{i0} + p \cdot y_i = e_{i0} + p \cdot e_i$. Solving for y_{i0} and substituting into the utility function, it follows that $x_{i0} = e_{i0} + p \cdot (e_i - x_i)$ and x_i solves

$$\begin{aligned} &\text{maximize} && \phi_i(y_i) + p \cdot (e_i - y_i) + e_{i0} \\ &\text{subject to} && y_i \ge 0. \end{aligned} \tag{4.4}$$

By the Inada condition, the nonnegativity constraint does not bind. Since e_{i0} is just a constant, dropping it and adding across i, it follows that $(x_i)_{i=1}^I$ solves

$$\text{maximize} \ \sum_{i=1}^I [\phi_i(y_i) + p \cdot (e_i - y_i)].$$

But this objective function is the Lagrangian of the optimization problem (4.3) (with Lagrange multiplier p), so by the KKT theorem $(x_i)_{i=1}^I$ solves (4.3).

Conversely, suppose that $(x_i)_{i=1}^I$ solves (4.3). Let p be the Lagrange multiplier and

$$\sum_{i=1}^I \phi_i(y_i) + p \cdot \sum_{i=1}^I (e_i - y_i)$$

[1] http://en.wikipedia.org/wiki/Jeremy_Bentham

be the Lagrangian. By the first-order condition with respect to y_i, we get

$$\nabla \phi_i(x_i) = p.$$

But this is the first-order condition of (4.4) (where $y_i \geq 0$ is not binding). Letting $x_{i0} = e_{i0} + p \cdot (e_i - x_i)$, by the KKT theorem (x_{i0}, x_i) solves the utility maximization problem. Since $\sum_{i=1}^I (x_i - e_i) \leq 0$, markets for goods $1, \dots, L$ clear. Since $p \cdot \sum_{i=1}^I (x_i - e_i) = 0$ by complementary slackness, it follows that $\sum_{i=1}^I (x_{i0} - e_{i0}) = 0$, so the market for the numéraire good clears. Therefore $\left\{(1, p), (x_{i0}, x_i)_{i=1}^I\right\}$ is a Walrasian equilibrium.

Since ϕ_i's are continuous and the constraint set in (4.3) is compact, the optimization problem (4.3) has a solution. Therefore an equilibrium always exists. Finally, if all ϕ_i's are strictly concave, so is $\sum_{i=1}^I \phi_i$, so the solution is unique. $\square$

The equilibrium existence (and uniqueness) theorem for a quasi-linear economy is simple and elegant: all we need to do is to add (the non-numéraire part of) individual utilities and maximize it, which is the mathematical formulation of Bentham's greatest happiness principle.

Chapter 5

Welfare properties of equilibrium

In this chapter we ask whether the market mechanism is desirable in some sense. To answer the question, we must first define what is "desirable". The minimal requirement is Pareto efficiency.

5.1 Pareto efficiency

Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. An allocation $(x_i)_{i=1}^I$ is said to be *feasible* if aggregate demand is less than or equal to aggregate endowment, that is,

$$\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i.$$

By having the inequality, we are allowing agents to throw away unnecessary goods for free. This property is called *free disposal.* Of course in the real world it costs something to get rid of garbage, a junk car, or nuclear waste, but this is a simplification.

Let $x = (x_i)_{i=1}^I$ and $y = (y_i)_{i=1}^I$ be two potential allocations. We would like to compare these two allocations and determine which is more socially desirable. One problem is that people have different opinions and it is difficult to agree. Hence we give up comparing any two allocations, and rank two allocations only when we get to unanimous consent, which leads to the definition of Pareto dominance.

Definition 5.1 (Pareto dominance)**.** An allocation (y_i) *Pareto dominates* the allocation (x_i) if $u_i(y_i) \geq u_i(x_i)$ for all i and $u_i(y_i) > u_i(x_i)$ for some i.

That is, an allocation Pareto dominates another if everybody is as well off and at least somebody is better off. We say that (y_i) weakly Pareto dominates (x_i) if $u_i(y_i) \geq u_i(x_i)$ for all i (without requiring a strict inequality). Similarly, we say that (y_i) strictly Pareto dominates (x_i) if $u_i(y_i) > u_i(x_i)$ for all i.

We now define the concept of Pareto efficiency. We say that an allocation (x_i) is Pareto inefficient if it is dominated by another feasible allocation (y_i). Otherwise, we say that it is efficient.

Definition 5.2 (Pareto efficiency)**.** Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. A feasible allocation (x_i) is said to be *Pareto efficient*, or just efficient, if there is no other feasible allocation (y_i) that Pareto dominates it.

That is, an allocation is efficient if you cannot make somebody better off without hurting somebody else. Pareto efficiency is sometimes called Pareto optimality. The concept of Pareto efficiency is named after Vilfredo Pareto.[1]

Pareto efficiency is a minimal requirement for "social good". If you can make somebody better off without hurting somebody else, why not do that? Making such an arrangement is called a Pareto improvement. A situation that can be Pareto improved is inefficient; a situation that is not inefficient is efficient, by definition.

Note that the notion of Pareto efficiency leaves out many issues, such as equity. For example, consider dividing a cake among a group of people, and suppose everybody likes eating the cake. Then giving the whole cake to one person and giving nothing to others is Pareto efficient. Similarly, dividing the cake equally is also Pareto efficient. Pareto efficiency merely states that resources are used efficiently, and the outcome can be equal or unequal.

5.2 First welfare theorem

The following theorem shows that a competitive equilibrium is Pareto efficient, which is a mathematical formulation of Adam Smith's "invisible hand":[2]

> As every individual, therefore, endeavours as much as he can both to employ his capital in the support of domestic industry, and so to direct that industry that its produce may be of the greatest value; every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it. By preferring the support of domestic to that of foreign industry, he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other eases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.

Smith's paragraph is very verbose, perhaps because it was written in the 18th century. His points can be summarized as follows:

- When people pursue their self interest, they promote social welfare as if led by an invisible hand.
- But promoting social welfare was not intentional.
- Free markets can promote social welfare better than a well-intended government.

[1] http://en.wikipedia.org/wiki/Vilfredo_Pareto
[2] http://en.wikipedia.org/wiki/Adam_smith

In short, in a market economy the pursuit of self-interest is socially beneficial, and government regulations are either unnecessary or even harmful.

Theorem 5.3 (First Welfare Theorem). *Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with locally nonsatiated utilities and $\{p, (x_i)\}$ be a Walrasian equilibrium. Then (x_i) is Pareto efficient.*

The proof is remarkably simple.

Proof. Suppose to the contrary that (x_i) is inefficient. Then there exists a feasible allocation (y_i) that Pareto dominates (x_i). By definition, we have $u_i(y_i) \geq u_i(x_i)$ for all i and $u_i(y_i) > u_i(x_i)$ for some i.

Consider an agent with $u_i(y_i) \geq u_i(x_i)$. Let us show that $p \cdot y_i \geq p \cdot e_i$. Suppose not. Then $p \cdot y_i < p \cdot e_i$. Since u_i is locally nonsatiated, for any $\epsilon > 0$ we can take y_i' such that $\|y_i' - y_i\| < \epsilon$ and $u_i(y_i') > u_i(y_i)$. Therefore $u_i(y_i') > u_i(x_i)$. By choosing sufficiently small $\epsilon > 0$, we may assume $p \cdot y_i' < p \cdot e_i$. Therefore y_i' is affordable but gives higher utility than x_i, which contradicts utility maximization. Therefore $p \cdot y_i \geq p \cdot e_i$ for all i.

Consider an agent with $u_i(y_i) > u_i(x_i)$. Let us show $p \cdot y_i > p \cdot e_i$. Suppose not. Then $p \cdot y_i \leq p \cdot e_i$, so y_i is affordable but gives higher utility than x_i, which contradicts utility maximization. Therefore $p \cdot y_i > p \cdot e_i$ for some i.

Summing these inequalities across i and noting that (y_i) is feasible, we obtain

$$\begin{aligned}
p \cdot \sum_{i=1}^{I} y_i &= \sum_{i=1}^{I} p \cdot y_i && (\because \text{exchange sum and inner product}) \\
&> \sum_{i=1}^{I} p \cdot e_i && (\because p \cdot y_i \geq p \cdot e_i, \text{ with at least one } >) \\
&= p \cdot \sum_{i=1}^{I} e_i && (\because \text{exchange sum and inner product}) \\
&\geq p \cdot \sum_{i=1}^{I} y_i && (\because \text{feasibility and } p \geq 0),
\end{aligned}$$

which is a contradiction. Therefore (x_i) is efficient. □

One of the first formal proofs of the first welfare theorem—which states that the market mechanism achieves an efficient allocation of resources—was, ironically, due to the socialist economist and Polish diplomat Oskar Lange.[3] His proof used complicated calculus and unnecessary assumptions (Lange, 1942). The above elegant proof was made independently by Arrow (1951) and Debreu (1951).

5.3 Second welfare theorem

The first welfare theorem states that the market mechanism achieves an efficient allocation of resources. Next we ask the converse: can any efficient allocation be achieved as a competitive equilibrium? To answer this question we need to define an equilibrium with transfer payments.

[3] http://en.wikipedia.org/wiki/Oskar_Lange

Definition 5.4. Let $\{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. A price p, an allocation (x_i), and transfer payments (t_i) constitute a *competitive equilibrium with transfer payments* if

(i) (Agent optimization) for each i, x_i solves

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \leq p \cdot e_i - t_i, \end{aligned} \tag{5.1}$$

(ii) (Market clearing) $\sum_{i=1}^I x_i \leq \sum_{i=1}^I e_i$,

(iii) (Balanced budget) $\sum_{i=1}^I t_i = 0$.

If $t_i > 0$, agent i *pays* a lump sum tax. If $t_i < 0$, agent i *receives* a lump sum transfer. The following theorem shows that (under reasonable assumptions) any Pareto efficient allocation is an equilibrium with transfer payments. The implication is that in order to achieve a specific Pareto efficient allocation, the government should not regulate markets but simply impose lump sum taxes, make lump sum transfers, and *laissez faire*.

Theorem 5.5 (Second Welfare Theorem)**.** *Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with continuous, quasi-concave, locally nonsatiated utilities. If (x_i) is a feasible Pareto efficient allocation with $x_i \gg 0$ for all i, then there exist a price vector p and transfer payments (t_i) such that $\{p, (x_i), (t_i)\}$ is a competitive equilibrium with transfer payments.*

Proof. Since (x_i) is feasible, market clearing automatically holds.

Let $U_i = \left\{ y \in \mathbb{R}^L_+ \,\middle|\, u_i(y) > u_i(x_i) \right\}$ be the set of consumption bundles that give higher utility to agent i than x_i (upper contour set) and

$$U = \sum_{i=1}^I U_i := \left\{ y = \sum_{i=1}^I y_i \,\middle|\, (\forall i) y_i \in U_i \right\}.$$

Let $E = \left\{ x \in \mathbb{R}^L \,\middle|\, x \leq \sum_{i=1}^I e_i \right\}$ be the set of vectors (not necessarily positive) less than or equal to the aggregate endowment. Since u_i is a locally nonsatiated quasi-concave function, U_i is a nonempty convex set, and so is U. Clearly E is a nonempty convex set. Since the allocation (x_i) is Pareto efficient, we have $U \cap E = \emptyset$. Therefore by the separating hyperplane theorem, there exists a nonzero vector p such that

$$(\forall x \in E)(\forall y \in U)\ p \cdot x \leq p \cdot y. \tag{5.2}$$

If $p_l < 0$ for some l, letting $x_l \to -\infty$ we get $p \cdot x > p \cdot y$, a contradiction. Therefore $p_l \geq 0$ for all l and $p > 0$.

Define the transfer t_i to satisfy the budget constraint, so

$$p \cdot x_i = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i).$$

Since (x_i) is feasible, we have $\sum_{i=1}^I (x_i - e_i) \leq 0$, so

$$\sum_{i=1}^I t_i = \sum_{i=1}^I p \cdot (e_i - x_i) = p \cdot \sum_{i=1}^I (e_i - x_i) \geq 0.$$

Setting $y = \sum_{i=1}^I y_i$ and $x = \sum_{i=1}^I e_i$ in (5.2) and letting $y_i \to x_i$ (which is possible by the local nonsatiation of u_i), we obtain

$$p \cdot \sum_{i=1}^I e_i \le p \cdot \sum_{i=1}^I x_i \iff \sum_{i=1}^I t_i = \sum_{i=1}^I p \cdot (e_i - x_i) \le 0.$$

Therefore $\sum_{i=1}^I t_i = 0$, so balanced budget holds.

To show that $\{p, (x_i), (t_i)\}$ is a competitive equilibrium with transfer payments, it remains to show agent optimization—that x_i solves the utility maximization problem (5.1). To this end take any $y_i \in U_i$ for each i. It suffices to show that $p \cdot y_i > p \cdot e_i - t_i$, for in that case any bundle y_i preferred to x_i is not affordable, which means that x_i gives highest utility within the budget set.

Assume $p \cdot y_i \le p \cdot e_i - t_i$ for some i. Without loss of generality we may assume $i = 1$. Since $x_1 \gg 0$ and $p > 0$, by the definition of t_1 we have

$$p \cdot e_1 - t_1 = p \cdot x_1 > 0.$$

Since $u_1(y_1) > u_1(x_1)$ and u_1 is continuous, there exists z_1 such that $u_1(z_1) > u_1(x_1)$ and $p \cdot z_1 < p \cdot x_1$. Let $\epsilon = p \cdot x_1 - p \cdot z_1 > 0$. By local nonsatiation, for every $i \ne 1$ there exists z_i such that $u_i(z_i) > u_i(x_i)$ and $p \cdot z_i \le p \cdot x_i + \epsilon/I$. By the definition of U_i, we have $z_i \in U_i$ for all i. Letting $y = \sum_{i=1}^I z_i$ and $x = \sum_{i=1}^I e_i$ in (5.2), it follows that

$$\begin{aligned}
\sum_{i=1}^I p \cdot e_i &= p \cdot \sum_{i=1}^I e_i \le p \cdot \sum_{i=1}^I z_i = \sum_{i=1}^I p \cdot z_i \\
&\le (p \cdot x_1 - \epsilon) + \sum_{i=2}^I (p \cdot x_i + \epsilon/I) = \sum_{i=1}^I p \cdot x_i - \frac{\epsilon}{I} \\
\implies \frac{\epsilon}{I} &\le \sum_{i=1}^I p \cdot (x_i - e_i) = -\sum_{i=1}^I t_i = 0,
\end{aligned}$$

which is a contradiction. Therefore $p \cdot y_i > p \cdot e_i - t_i$ for all i. □

5.4 Characterizing Pareto efficient allocations

To apply the second welfare theorem, one needs to start with a Pareto efficient allocation. Therefore the theorem would be vacuous unless we can characterize the set of Pareto efficient allocations. This section provides a solution based on the marginal rate of substitution.

Let u be a differentiable utility function with $\nabla u \gg 0$. The marginal rate of substitution (MRS) between goods l and m is the ratio of marginal utilities,

$$\mathrm{MRS}_{lm} = \frac{\partial u / \partial x_l}{\partial u / \partial x_m}.$$

We can show that an allocation is Pareto efficient if and only if the marginal rate of substitution is equal across agents.

Note that if agents are maximizing utility subject to budget constraints, by the first-order condition there exists $\lambda_i > 0$ such that $\nabla u_i(x_i) = \lambda_i p$, where p

is the price vector. Therefore agent i's marginal rate of substitution between goods l, m is

$$\mathrm{MRS}_{i,lm} = \frac{\partial u_i/\partial x_l}{\partial u_i/\partial x_m} = \frac{\lambda_i p_l}{\lambda_i p_m} = \frac{p_l}{p_m},$$

which does not depend on i. Therefore MRS is equalized across agents. Conversely, if MRS is the same for all agents, in particular

$$p_l := \frac{\partial u_i/\partial x_l}{\partial u_i/\partial x_1}$$

does not depend on i. Letting $\lambda_i := \partial u_i/\partial x_1$, we obtain $\partial u_i/\partial x_l = \lambda_i p_l$, so $\nabla u_i = \lambda_i p$ for the price vector $p = (1, p_2, \dots, p_L)'$.

Thus MRS is equalized across agents if and only if the vectors of marginal utilities ∇u_i are collinear (point to the same direction p). With this property in mind, we can show the following results.

Proposition 5.6. *Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with quasi-concave utilities such that $\nabla u_i \gg 0$. Let (x_i) be an allocation such that $x_i \gg 0$ for all i and $\sum_{i=1}^I x_i = \sum_{i=1}^I e_i$. Then (x_i) is Pareto efficient if and only if the marginal rate of substitution is equalized across agents.*

Proof. Suppose (x_i) is Pareto efficient. Since u_i is differentiable, it is continuous. Since $\nabla u_i \gg 0$, u_i is strongly monotonic, and in particular locally nonsatiated. Since $x_i \gg 0$ for all i, by the second welfare theorem there exist a price vector p and transfers (t_i) such that $\{p, (x_i), (t_i)\}$ is a competitive equilibrium with transfer payments. Let

$$L_i = u_i(x) + \lambda_i(p \cdot e_i - t_i - p \cdot x_i)$$

be the Lagrangian of agent i's utility maximization problem, where we ignore the nonnegativity constraint $x \geq 0$ because we know $x_i \gg 0$ by assumption. By the Karush-Kuhn-Tucker theorem, there exists $\lambda_i \geq 0$ such that $\nabla u_i(x_i) = \lambda_i p$ for all i. Since $\nabla u_i \gg 0$, it must be $\lambda_i > 0$ and $p \gg 0$, and so MRS is equal across agents.

Conversely, suppose that MRS is equal across agents. By the above argument, we can take $p \gg 0$ and $\lambda_i > 0$ such that $\nabla u_i(x_i) = \lambda_i p$ for all i. Let

$$L_i = u_i(x) + \lambda_i(p \cdot x_i - p \cdot x)$$

be the Lagrangian of the utility maximization problem

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \leq p \cdot x_i \end{aligned}$$

with initial endowment x_i. Since $0 \ll \nabla u_i(x_i) = \lambda_i p$, by the sufficiency of KKT conditions for quasi-concave maximization, x_i is a solution to the utility maximization problem. Therefore if $u_i(y_i) > u_i(x_i)$, it must be $p \cdot y_i > p \cdot x_i$.

Now suppose that a feasible allocation (y_i) Pareto dominates (x_i). Since utilities are strongly monotonic ($\nabla u_i \gg 0$), without loss of generality we may assume that $u_i(y_i) > u_i(x_i)$, so $y_i \in U_i$. (See Problem 5.1.) Therefore $p \cdot y_i > p \cdot x_i$. Adding across agents, we obtain

$$p \cdot \sum_{i=1}^I e_i \geq p \cdot \sum_{i=1}^I y_i > p \cdot \sum_{i=1}^I x_i = p \cdot \sum_{i=1}^I e_i,$$

which is a contradiction. ∎

Notes

The content of this chapter is roughly the same as Chapter 19 of Starr (2011). That chapter explains how to relax the assumption $x_i \gg 0$.

Exercises

5.1. Let $\mathcal{E} = \{I, (u_i), (e_i)\}$ be an Arrow-Debreu economy with continuous, strongly monotonic utility functions. Show that if (x_i) is a Pareto inefficient allocation, then there exists a feasible allocation (y_i) such that $u_i(y_i) > u_i(x_i)$ for all i.

5.2. Consider an economy with two agents and two goods. The endowments are $e_1 = (1, 9)$ and $e_2 = (9, 1)$. The utility functions are

$$\begin{aligned} u_1(x_1, x_2) &= x_1 x_2, \\ u_2(x_1, x_2) &= \min\{x_1 x_2, 16\}. \end{aligned}$$

(i) Show that the price vector $p = (1, 1)$ and the allocation $x_1 = x_2 = (5, 5)$ constitute an equilibrium.

(ii) Show that the above equilibrium is Pareto inefficient.

(iii) Does this example contradict the first welfare theorem? Explain.

5.3. Consider an economy with two agents and two goods with Cobb-Douglas utilities

$$\begin{aligned} u_1(x_1, x_2) &= \alpha \log x_1 + (1 - \alpha) \log x_2, \\ u_2(x_1, x_2) &= \beta \log x_1 + (1 - \beta) \log x_2, \end{aligned}$$

where $0 < \alpha, \beta < 1$. Suppose that the aggregate endowment is $e = (e_1, e_2)$. Find all Pareto efficient allocations such that each agent consumes a positive amount of each good.

5.4. Let $\mathcal{E} = \{I, J, (e_i), (u_i), (Y_j), (\theta_{ij})\}$ be an Arrow-Debreu economy with production.

(i) Define the concept of Pareto efficiency.

(ii) Show that if u_i's are locally nonsatiated and $\{p, (x_i), (y_j)\}$ is a competitive equilibrium, then $\{(x_i), (y_j)\}$ is Pareto efficient.

5.5. Let $\mathcal{E} = \{I, J, (e_i), (u_i), (Y_j), (\theta_{ij})\}$ be an Arrow-Debreu economy with production.

(i) Define a competitive equilibrium with transfer payments.

(ii) Show that if u_i's are continuous, quasi-concave, locally nonsatiated, Y_j's are convex, and $\{(x_i), (y_j)\}$ is a feasible Pareto efficient allocation such that $x_i \gg 0$ for all i, then there exists a price vector p and transfer payments (t_i) such that $\{p, (x_i), (y_j), (t_i)\}$ is a competitive equilibrium with transfer payments.

5.6. Consider an economy with I agents and L basic goods labeled by $l = 1, \dots, L$. Suppose that there is another good, labeled 0, which is a public good. (A public good is non-excludable, i.e., the consumption of one agent does not reduce the availability of that good to other agents. Therefore all agents consume the same amount of good 0, which equals aggregate supply in equilibrium.) Suppose that there are no endowments of good 0, which is produced from other goods $l = 1, \dots, L$ using some technology represented by a production function $y = f(x_1, \dots, x_L)$. It is well known that the presence of a public good may make the economy inefficient. Let $u_i(x_0, x_1, \dots, x_L)$ be the utility function of agent i, assumed to be locally nonsatiated.

(i) Show that by quoting an individual-specific price for the public good, we can make the competitive equilibrium allocation efficient. (Hint: expand the set of goods, and consider an economy with $L + I$ goods labeled by $l = 1, \dots, L+I$. Goods $l = 1, \dots, L$ are the L basic goods, and good $L+i$ is the public good consumed by agent i. Make sure to discuss how we should reinterpret the production technology and market clearing conditions.)

(ii) Suppose that there are two agents ($i = 1, 2$), one basic good, and a public good. Agent i has utility function

$$u_i(x_0, x_1) = \alpha_i \log x_0 + (1 - \alpha_i) \log x_1,$$

where x_0, x_1 are consumption of the public good and the basic good. Let e_i be the initial endowment of agent i's basic good, and suppose there is a technology that converts the basic good to the public good one-for-one. Normalize the price of the basic good to be 1. Find individual-specific prices for the public good to make the competitive equilibrium allocation efficient.

Chapter 6

Existence of equilibrium

So far we have studied the welfare properties of the Arrow-Debreu equilibrium, assuming its existence. This is a dangerous avenue, for it makes no sense to study something unless it exists. Once a mathematics professor told me a funny story. He had a doctoral student who studied the properties of some set. The student developed elaborate arguments and proved theorems, but without constructing an explicit example. When the professor asked the student for an example, he worked hard to make one, and found that the only set that satisfied his theory was a set consisting of a single point. A theory of a single point is not so exciting.

So the natural question is, does an Arrow-Debreu equilibrium always exist? The answer is yes, under reasonable assumptions. Below I describe the idea of the proof and explain the technical difficulties. Then I rigorously prove the existence of equilibrium.

6.1 Idea of the proof and difficulties

The idea to prove the existence of equilibrium is to apply a fixed point theorem. The well-known Brouwer fixed point theorem is particularly easy to remember.

Brouwer fixed point theorem. *Let $C \subset \mathbb{R}^L$ be a nonempty, compact, convex set. Let $f : C \to C$ be continuous. Then f has a fixed point, i.e., there exists $x \in C$ such that $f(x) = x$.*

The Brouwer fixed point theorem is intuitive: if you stir coffee in a coffee cup, you will see it spin around a point (or points). This point is a fixed point of the (instantaneous) movement of coffee. I will not prove the theorem because it is difficult. Starr, 2011 contains an accessible proof. However, I explain why each assumption is necessary. Remember that a compact set in $\mathbb{R}^L$ is closed and bounded.

Example 6.1 (Necessity of closedness)**.** Let $C = (0, 1]$ and $f(x) = x/2$. Then C is nonempty, bounded, convex, and $f : C \to C$ is continuous, but f has no fixed point.

Example 6.2 (Necessity of boundedness)**.** Let $C = \mathbb{R}$ and $f(x) = x + 1$. Then C is nonempty, closed, convex, and $f : C \to C$ is continuous, but f has no fixed point.

Example 6.3 (Necessity of convexity)**.** Let $C = S^1 = \left\{x \in \mathbb{R}^2 \,\middle|\, \|x\| = 1\right\}$ and $f(x) = -x$. Then C is nonempty, compact (closed and bounded), and $f : C \to C$ is continuous, but f has no fixed point.

Example 6.4 (Necessity of continuity)**.** Let $C = [0, 1]$ and $f(x) = 1$ if $x < 1/2$ and $f(x) = 0$ if $x \geq 1/2$. Then C is nonempty, compact, and convex, but f has no fixed point.

Let us turn to the existence of equilibrium. Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. Remember that an equilibrium consists of a price p and allocation (x_i) such that each x_i solves the utility maximization problem given p and the markets clear. Let $x_i(p, w)$ be the demand of agent i with wealth w when the price is p. Let

$$z(p) = \sum_{i=1}^{I} (x_i(p, p \cdot e_i) - e_i)$$

be the excess demand, where $p \cdot e_i$ is the wealth of agent i. Then we have an equilibrium if we find a price p such that $z(p) \leq 0$. Note that by the neutrality of money the price level does not matter. So without loss of generality, we may assume that the price vector p belongs to the simplex

$$\Delta^{L-1} = \left\{p \in \mathbb{R}^L_+ \,\middle|\, p_1 + \cdots + p_L = 1\right\}.$$

The following theorem shows that any continuous function defined on the simplex and satisfying Walras' law can be made to have all coordinates nonpositive. The idea of the proof is to raise the price of a good in excess demand.

Theorem 6.1. *Let $z : \Delta^{L-1} \to \mathbb{R}^L$ be continuous and $p \cdot z(p) = 0$ for all $p \in \Delta^{L-1}$. Then there exists p such that $z(p) \leq 0$.*

Proof. Define $f : \Delta^{L-1} \to \Delta^{L-1}$ by

$$f_l(p) = \frac{p_l + \max\{0, z_l(p)\}}{1 + \sum_{l=1}^{L} \max\{0, z_l(p)\}}.$$

That is, we increase the price of good l to $p_l + z_l(p)$ if $z_l(p) > 0$, leave it unchanged if $z_l(p) \leq 0$, and normalize so that it stays in Δ^{L-1}. Since z is continuous, so is f. Therefore by the Brouwer fixed point theorem there exists p such that $f(p) = p$, so

$$p_l = \frac{p_l + \max\{0, z_l(p)\}}{1 + \sum_{l=1}^{L} \max\{0, z_l(p)\}}$$

for all l. Multiplying both sides by $z_l(p)$ and using the Walras law, we get

$$\begin{aligned} 0 = \sum_{l=1}^{L} p_l z_l(p) &= \frac{\sum_{l=1}^{L} (p_l z_l(p) + z_l(p) \max\{0, z_l(p)\})}{1 + \sum_{l=1}^{L} \max\{0, z_l(p)\}} \\ \iff \sum_{l=1}^{L} z_l(p) \max\{0, z_l(p)\} &= 0. \end{aligned}$$

Since

$$z_l(p)\max\{0, z_l(p)\} = \begin{cases} 0, & (z_l(p) \le 0) \\ z_l(p)^2 > 0, & (z_l(p) > 0) \end{cases}$$

it must be $z_l(p) \le 0$ for all l. Therefore $z(p) \le 0$. □

Theorem 6.1 cannot directly be applied to prove the existence of a competitive equilibrium, however. There are a few difficulties. First, unless the utility functions are strictly quasi-concave, individual demand is not necessarily a single point. In general, demand consists of a set. This issue is not difficult to overcome by using the Kakutani fixed point theorem, which generalizes the Brouwer fixed point theorem to the case of correspondences. I will return to this point later. Second, in Theorem 6.1 we assumed that the excess demand function is continuous, but this should be proved from more basic assumptions. If agents' incomes are strictly positive, we can prove continuity using the maximum theorem, which I will mention later. Finally, we have to allow the possibility of free goods like air, which has price zero. However, typically when the price is zero, agents will demand an infinite amount of the good, so the excess demand is not well-defined. This is the most difficult issue to overcome.

Can we prove the existence of equilibrium by a more elementary method, without invoking the fixed point theorem? Interestingly, the answer is no. The following proposition shows that if an excess demand function satisfying the Walras law can always clear the markets, then we can prove the Brouwer fixed point theorem. Essentially, existence of equilibrium and the Brouwer fixed point theorem are equivalent.

Proposition 6.2 (Uzawa, 1962)**.** *Consider the following two statements:*

A. For any continuous function $z : \Delta^{L-1} \to \mathbb{R}^L$ satisfying $p \cdot z(p) = 0$, there exists $p \in \Delta^{L-1}$ such that $z(p) \le 0$.

B. Any continuous function $f : \Delta^{L-1} \to \Delta^{L-1}$ has a fixed point.

Then A implies B.

Proof. Suppose statement A is true. Let $f : \Delta^{L-1} \to \Delta^{L-1}$ be continuous and $g(p)$ be the projection of $f(p)$ on the line $\{tp \mid t \in \mathbb{R}\}$ (Figure 6.1). Letting $g(p) = tp$, we obtain

$$\langle f(p) - tp, p\rangle = 0 \iff t = \frac{\langle p, f(p)\rangle}{\|p\|^2},$$

so $g(p) = \frac{\langle p, f(p)\rangle}{\|p\|^2} p$. Define

$$z(p) = f(p) - g(p) = f(p) - \frac{\langle p, f(p)\rangle}{\|p\|^2} p.$$

Then $z : \Delta^{L-1} \to \mathbb{R}^L$ is continuous and $p \cdot z(p) = 0$. Therefore by assumption there exists $p \in \Delta^{L-1}$ such that $z(p) \le 0$, so $f_l(p) \le \lambda p_l$ for all l, where $\lambda = \frac{\langle p, f(p)\rangle}{\|p\|^2}$. Then

$$\lambda = \frac{\langle p, f(p)\rangle}{\|p\|^2} \le \frac{\langle p, \lambda p\rangle}{\|p\|^2} = \lambda,$$

so either $p_l = 0$ or $f_l(p) = \lambda p_l$ for all l. Since $f_l(p) \geq 0$, in either case $f_l(p) = \lambda p_l$ for all l. Adding across l we get

$$1 = \sum_{l=1}^{L} f_l(p) = \lambda \sum_{l=1}^{L} p_l = \lambda,$$

so $f_l(p) = p_l$. This shows that $f(p) = p$, so p is a fixed point of f. □

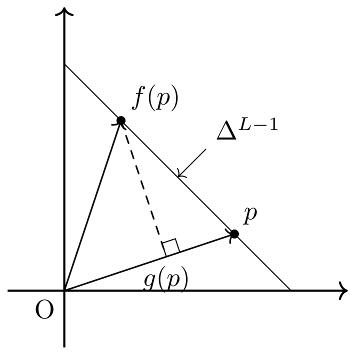

Figure 6.1: Construction of $g(p)$.

6.2 Correspondence and the maximum theorem

In this section I introduce mathematical concepts that are needed to prove the existence of equilibrium. Two theorems will play a crucial role: the Kakutani fixed point theorem and the maximum theorem. Both involve correspondences, or point-to-set mappings.

Let $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ be sets. $\Gamma : X \twoheadrightarrow Y$ is a *correspondence* (or *multi-valued function*) if for each $x \in X$ we have $\Gamma(x) \subset Y$, a subset of Y. Note that we use an arrow with two heads "$\twoheadrightarrow$" for a correspondence, while we use the usual arrow "$\to$" for a function. Another common notation is $\Gamma : X \rightrightarrows Y$. Γ is said to be *compact (convex) valued* if for each $x \in X$, the set $\Gamma(x)$ is compact (convex). Γ is said to be *uniformly bounded* if for each $\bar{x} \in X$, there exists a neighborhood U of $\bar{x}$ such that $\bigcup_{x \in U} \Gamma(x)$ is bounded. Of course, $\Gamma(x)$ need not be uniformly bounded just because $\Gamma(x)$ is bounded. For instance, let $X = \mathbb{R}$ and

$$\Gamma(x) = \begin{cases} [0, 1], & (x \leq 0) \\ [0, 1/x]. & (x > 0) \end{cases}$$

Then $\Gamma(x)$ is bounded but not uniformly bounded at $\bar{x} = 0$ (draw a picture).

Remember that a function $f : X \to Y$ is continuous if $x_n \to x$ implies $f(x_n) \to f(x)$. "$x_n \to x$" is a shorthand notation for "$\lim_{n\to\infty} x_n = x$". We can define continuity for correspondences.

Definition 6.3 (Upper hemicontinuity)**.** $\Gamma : X \twoheadrightarrow Y$ is *upper hemicontinuous* if it is uniformly bounded and $x_n \to x$, $y_n \in \Gamma(x_n)$, and $y_n \to y$ implies $y \in \Gamma(x)$.

Upper hemicontinuity is also called upper semi-continuity (I often use semi-continuity). Perhaps hemicontinuity is less confusing since there is a separate semi-continuity concept for functions, introduced below. When the requirement that Γ is uniformly bounded is dropped, then Γ is called *closed*. When Y is itself bounded, upper hemicontinuity is the same as closedness. Upper hemicontinuity says that if a sequence in the image of a convergent sequence is convergent, then the limit belongs to the image of the limit. There is also a concept called lower hemicontinuity, which is roughly the converse. If you take a point in the image of the limit, then you can take a sequence in the image of the sequence that converges to that point.

Definition 6.4 (Lower hemicontinuity)**.** $\Gamma : X \twoheadrightarrow Y$ is *lower hemicontinuous* if for any $x_n \to x$ and $y \in \Gamma(x)$, there exists a number N and a sequence $y_n \to y$ such that $y_n \in \Gamma(x_n)$ for $n > N$.

A correspondence that is both upper and lower hemicontinuous is called *continuous*. For upper hemicontinuous correspondences, we have the following Kakutani fixed point theorem.

Kakutani fixed point theorem. *Let $C \subset \mathbb{R}^L$ be a nonempty, compact, convex set. Let $\Gamma : C \twoheadrightarrow C$ be a nonempty, convex valued upper hemicontinuous correspondence. Then Γ has a fixed point, i.e., there exists $x \in C$ such that $x \in \Gamma(x)$.*

The Brouwer fixed point theorem is a special case of the Kakutani fixed point theorem by setting $\Gamma(x) = \{f(x)\}$, a correspondence consisting of a single point. The proof of the Kakutani fixed point theorem is difficult. See Berge (1963).

The next maximum theorem guarantees that the solution set of a parametric maximization problem is upper hemicontinuous, which allows us to apply the Kakutani fixed point theorem.

Maximum theorem. *Let $f : X \times Y \to \mathbb{R}$ and $\Gamma : X \twoheadrightarrow Y$ be continuous. Assume*

$$\Gamma^*(x) = \arg\max_{y \in \Gamma(x)} f(x, y) \neq \emptyset$$

and let $f^*(x) = \max_{y \in \Gamma(x)} f(x, y)$. Then f^* is continuous and $\Gamma^* : X \twoheadrightarrow Y$ is upper hemicontinuous.

The proof of the maximum theorem is not so difficult, but it is clearer to weaken the assumptions and prove several weaker statements. To do so I define semi-continuity for functions.

Definition 6.5 (Semi-continuity of functions)**.** $f : X \to [-\infty, \infty]$ is *upper semi-continuous* if $x_n \to x$ implies $\limsup_{n \to \infty} f(x_n) \leq f(x)$. f is *lower semi-continuous* if $x_n \to x$ implies $\liminf_{n \to \infty} f(x_n) \geq f(x)$.

Clearly, f is upper semi-continuous if $-f$ is lower semi-continuous, and f is continuous if it is both upper and lower semi-continuous. It is well-known that a continuous function attains the maximum on a compact set. Indeed all we need is upper semi-continuity, as the following proposition shows.

Proposition 6.6. *Let X be nonempty and compact and $f : X \to [-\infty, \infty)$ upper semi-continuous. Then f attains the maximum on X.*

Proof. Let $M = \sup_{x\in X} f(x)$. Take $\{x_n\}$ such that $f(x_n) \to M$. Since X is compact, $\{x_n\}$ has a convergent subsequence. Assume $x_{n_k} \to x$. Since f is upper semi-continuous, we have

$$M \geq f(x) \geq \limsup_{k\to\infty} f(x_{n_k}) = M,$$

so $f(x) = M = \max_{x\in X} f(x)$. □

By the same argument, a lower semi-continuous function attains the minimum on a compact set. Proposition 6.6 is useful. For example, the Cobb-Douglas utility function

$$u(x_1, x_2) = \alpha_1 \log x_1 + \alpha_2 \log x_2$$

is not continuous at $x_1 = 0$ or $x_2 = 0$ in the usual sense. But if we define $\log 0 = -\infty$, u becomes upper semi-continuous. Therefore if the budget set is compact, we know a priori that a solution to the utility maximization problem exists.

We prove two lemmas to prove the maximum theorem.

Lemma 6.7. *Let $f : X\times Y \to \mathbb{R}$ be upper semi-continuous and $\Gamma : X \twoheadrightarrow Y$ upper hemicontinuous. Then $f^*(x) = \sup_{y\in\Gamma(x)} f(x, y)$ is upper semi-continuous.*

Proof. Take any $x_n \to x$ and $\epsilon > 0$. Take a subsequence $\{x_{n_k}\}$ such that $f^*(x_{n_k}) \to \limsup_{n\to\infty} f^*(x_n)$. For each k, take $y_{n_k} \in \Gamma(x_{n_k})$ such that $f(x_{n_k}, y_{n_k}) > f^*(x_{n_k}) - \epsilon$. Since Γ is upper hemicontinuous, it is uniformly bounded. Therefore there exists a neighborhood U of x such that $\bigcup_{x'\in U} \Gamma(x')$ is bounded. Since $x_{n_k} \to x$, there exists K such that $\bigcup_{k>K} \Gamma(x_{n_k})$ is bounded. Hence $\{y_{n_k}\}$ is bounded. By taking a subsequence if necessary, we may assume $y_{n_k} \to y$. Since Γ is upper hemicontinuous, we have $y \in \Gamma(x)$. Since f is upper semi-continuous,

$$f^*(x) \geq f(x, y) \geq \limsup_{k\to\infty} f(x_{n_k}, y_{n_k}) \geq \lim_{k\to\infty} f^*(x_{n_k}) - \epsilon = \limsup_{n\to\infty} f^*(x_n) - \epsilon.$$

Letting $\epsilon \to 0$, it follows that f^* is upper semi-continuous. □

Lemma 6.8. *Let $f : X\times Y \to \mathbb{R}$ be lower semi-continuous and $\Gamma : X \twoheadrightarrow Y$ lower hemicontinuous. Then $f^*(x) = \sup_{y\in\Gamma(x)} f(x, y)$ is lower semi-continuous.*

Proof. Take any $x_n \to x$ and $\epsilon > 0$. Take $y \in \Gamma(x)$ such that $f(x, y) > f^*(x) - \epsilon$. Since Γ is lower hemicontinuous, there exist N and $y_n \to y$ such that $y_n \in \Gamma(x_n)$ for all $n > N$. Then $f^*(x_n) \geq f(x_n, y_n)$. Since f is lower semi-continuous,

$$\liminf_{n\to\infty} f^*(x_n) \geq \liminf_{n\to\infty} f(x_n, y_n) \geq f(x, y) > f^*(x) - \epsilon.$$

Letting $\epsilon \to 0$, it follows that f^* is lower semi-continuous. □

Proof of the maximum theorem. By Lemmas (6.7) and (6.8), f^* is continuous. Since $\Gamma^*(x) \subset \Gamma(x)$ and Γ is uniformly bounded, so is Γ^*. Take any $x_n \to x$, $y_n \in \Gamma^*(x_n)$, and assume $y_n \to y$. Since f and f^* are continuous, we have

$$f(x, y) = \lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} f^*(x_n) = f^*(x),$$

so $y \in \Gamma^*(x)$. Hence Γ^* is upper hemicontinuous. □

6.3 Existence of equilibrium

Now we have all the ingredients to prove the existence of equilibrium.

Theorem 6.9 (Existence of equilibrium)**.** *Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. Assume that for all i, $e_i \gg 0$ and u_i is continuous, quasi-concave, and locally nonsatiated. Then $\mathcal{E}$ has a competitive equilibrium.*

The idea of the proof is to "truncate" the economy so that agents do not demand an infinite amount of a good when its price approaches zero, prove the existence of equilibrium in this truncated economy, and then argue that the truncation does not matter.

Proof.

Step 1. Construction of the truncated economy and the set to apply the Kakutani fixed point theorem.

Take $b > 0$ such that b is larger than the aggregate endowment of any good, so $\sum_{i=1}^{I} e_{il} < b$ for all l. Let

$$X_b = [0, b]^L = \left\{x \in \mathbb{R}_+^L \,\middle|\, (\forall l) x_l \leq b\right\}$$

be the set of consumption bundles bounded by b. Let $C = \Delta^{L-1} \times X_b^I$. Clearly C is nonempty, compact, and convex.

Step 2. Construction of the fixed point correspondence.

Define $\Gamma : C \twoheadrightarrow C$ as follows. Let $(p, (x_i)) \in C = \Delta^{L-1} \times X_b^I$. Define

$$\Gamma_0(p, (x_i)) = \operatorname*{arg\,max}_{q \in \Delta^{L-1}} q \cdot \sum_{i=1}^{I} (x_i - e_i),$$

that is, Γ_0 consists of price vectors that maximize the value of excess demand. Since Δ^{L-1} is nonempty, compact, convex, and the objective function is linear (hence continuous and quasi-concave), Γ_0 is nonempty, compact, and convex.

For each $i = 1, \dots, I$, define

$$\Gamma_i(p, (x_i)) = \operatorname*{arg\,max}_{x \in X_b \cap B_i(p)} u_i(x),$$

that is, Γ_i consists of consumption bundles that maximize utility within the budget constraint and the set X_b. Since $X_b \cap B_i(p)$ is nonempty, compact, convex, and the objective function is continuous and quasi-concave, Γ_i is nonempty, compact, and convex.

Define $\Gamma : C \twoheadrightarrow C$ by $\Gamma = \Gamma_0 \times \Gamma_1 \times \cdots \times \Gamma_I$.

Step 3. Existence of a fixed point.

Since C is nonempty, compact, convex, and Γ is nonempty, compact, convex valued, by the Kakutani fixed point theorem Γ has a fixed point if Γ is upper hemicontinuous. To show that Γ is upper hemicontinuous, it suffices to show that each of $\Gamma_0, \dots, \Gamma_I$ is upper hemicontinuous.

Since $q \cdot \sum_{i=1}^{I} (x_i - e_i)$ is continuous in $q, p, x_1, \dots, x_I$ and the set Δ^{L-1} is fixed (hence continuous), by the maximum theorem Γ_0 is upper hemicontinuous.

Since $u_i(x)$ is continuous in $x, p, x_1, \dots, x_I$, to show that Γ_i is upper hemicontinuous, by the maximum theorem it suffices to show that $p \mapsto X_b \cap B_i(p)$ is continuous. If $p_n \to p$, $x_n \in X_b \cap B_i(p_n)$, and $x_n \to x$, then by the definition of the budget constraint we have $p_n \cdot (x_n - e_i) \leq 0$. Letting $n \to \infty$, we get $p \cdot (x - e_i) \leq 0$, so $x \in X_b \cap B_i(p)$. Since X_b is compact, $p \mapsto X_b \cap B_i(p)$ is upper hemicontinuous.

To show that $p \mapsto X_b \cap B_i(p)$ is lower hemicontinuous, let $p_n \to p$ and $x \in X_b \cap B_i(p)$. Define $x_n = t_n x$, where

$$t_n = \min \left\{ \frac{p_n \cdot e_i}{p_n \cdot x}, 1 \right\}.$$

Graphically, x_n is x itself if x is affordable at price p_n (so $p_n \cdot x \leq p_n \cdot e_i$), and otherwise it is the intersection between the line connecting 0 and x and the budget line (Figure 6.2). Since $e_i \gg 0$, we have $p_n \cdot e_i > 0$ for all n, so t_n is well-defined even if $p_n \cdot x = 0$ by setting $p_n \cdot e_i / p_n \cdot x = \infty$. Since $x \in B_i(p)$, we have $p \cdot x \leq p \cdot e_i$, so $t_n \to 1$. Since by construction $x_n \in X_b \cap B_i(p_n)$ and $x_n = t_n x \to x$, it follows that $p \mapsto X_b \cap B_i(p)$ is lower hemicontinuous.

By the Kakutani fixed point theorem, there exists $(p, (x_i)) \in \Delta^{L-1} \times X_b^I$ such that $(p, (x_i)) \in \Gamma(p, (x_i))$.

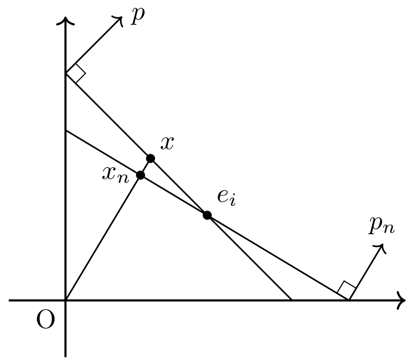

Figure 6.2: Definition of x_n.

Step 4. The allocation (x_i) is feasible.

Since $x_i \in B_i(p)$, we have $p \cdot (x_i - e_i) \leq 0$. Adding across i, we have $p \cdot \sum_{i=1}^I (x_i - e_i) \leq 0$. If $\sum_{i=1}^I (x_{il} - e_{il}) > 0$ for some l, setting $q_l = 1$ and zero for other goods, we obtain

$$q \cdot \sum_{i=1}^I (x_i - e_i) > 0 \geq p \cdot \sum_{i=1}^I (x_i - e_i),$$

which contradicts the fact that $p \in \Gamma_0(p, (x_i))$ and hence p maximizes the value of excess demand. Therefore $\sum_{i=1}^I (x_i - e_i) \leq 0$, so (x_i) is feasible.

Step 5. x_i solves the utility maximization problem.

If x_i does not solve the utility maximization problem for some i, then there exists x such that $p \cdot x \leq p \cdot e_i$ and $u_i(x) > u_i(x_i)$ (Figure 6.3). Since u_i is continuous and $p \cdot e_i > 0$ since $e_i \gg 0$, there exists x' such that $p \cdot x' < p \cdot e_i$ and $u_i(x') > u_i(x_i)$. Since (x_i) is feasible, we have $x_i \in [0, b)^L$. Therefore, for sufficiently small $\alpha > 0$ we have $x'' := (1-\alpha)x_i + \alpha x' \in [0, b)^L$. Since u_i is quasi-concave, we have

$$u_i(x'') \geq \min \{u_i(x_i), u_i(x')\} = u_i(x_i).$$

Since $p \cdot x_i \leq p \cdot e_i$, $p \cdot x' < p \cdot e_i$, and $\alpha > 0$, we have

$$p \cdot x'' = (1-\alpha)p \cdot x_i + \alpha p \cdot x' < p \cdot e_i.$$

Since u_i is locally nonsatiated, there exists $x''' \in X_b$ such that $p \cdot x''' < p \cdot e_i$ and $u_i(x''') > u_i(x'') \geq u_i(x_i)$, which contradicts $x_i \in \Gamma_i(p, (x_i))$. Therefore x_i solves the utility maximization problem.

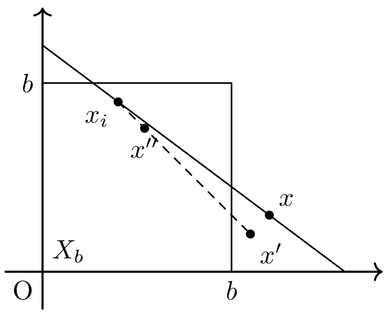

Figure 6.3: Definitions of x, x', x''.

This completes the proof that $\{p, (x_i)\}$ is a competitive equilibrium. □

Theorem 6.9 assumes that any good can be bought or sold in any nonnegative amount. This assumption is unrealistic, since for example nobody can consume more than 24 hours of leisure per day. (Working is interpreted as selling leisure.) However, this restriction is not essential. Theorem 6.9 also assumes $e_i \gg 0$ for all i, so all agents hold a tiny amount of all goods. This assumption is sufficient to guarantee that agents have positive wealth at all prices, which is necessary to show that the budget set is continuous in p. Clearly $e_i \gg 0$ is unrealistic since most of us don't have endowments of many goods, such as aircrafts. McKenzie, 1954 shows how to relax this assumption.

Notes

The content of this note is roughly the same as Chapters 18, 23, and 24 of Starr (2011). Chapter 9 contains a proof of the Brouwer fixed point theorem.

The existence of competitive equilibrium is generally credited to Arrow and Debreu (1954), but McKenzie (1954) published his proof one issue earlier. The major difference between these two papers is that McKenzie assumes the continuity of the demand function (which is an endogenous object) and ignores the issue with zero prices, whereas Arrow and Debreu prove the continuity from primitives. See Geanakoplos (1987), Duffie and Sonnenschein (1989), and Weintraub (2011) for historical accounts.

Chapter 7

Uniqueness of equilibrium

7.1 Sonnenschein-Mantel-Debreu theory

It would be nice if an economy has a unique equilibrium. To study this question, let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with strongly monotonic, strictly quasi-concave, and smooth utility functions. Given any price vector $p \gg 0$, then the demand $x_i(p, p \cdot e_i)$ is unique and continuous. Let

$$z(p) = \sum_{i=1}^{I} (x_i(p, p \cdot e_i) - e_i)$$

be the aggregate excess demand. Then z is continuous. By the neutrality of money, we have $z(tp) = z(p)$, so z is homogeneous of degree zero. By the Walras law, we have $p \cdot z(p) = 0$. Note that p is an equilibrium price vector if and only if $z(p) = 0$. Can we derive other properties of the aggregate excess demand function so that the equation $z(p) = 0$ has a unique solution (up to a multiplicative constant)?

Unfortunately, according to the results of Sonnenschein (1972, 1973), Mantel (1974), and Debreu (1974), "anything goes" for the aggregate excess demand function. More precisely, we have the following result.

Theorem 7.1 (Debreu (1974))**.** *Let* $0 < \epsilon < 1/L$ *and*

$$\Delta_\epsilon = \left\{ p \in \mathbb{R}^L \,\middle|\, \sum_{l=1}^{L} p_l = 1, (\forall l) p_l \geq \epsilon \right\}$$

be the set of normalized prices bounded below by $\epsilon > 0$. *Let* $f : \Delta_\epsilon \to \mathbb{R}^L$ *be continuous and satisfy* $p \cdot f(p) = 0$. *Then there exists an economy with* L *agents such that the aggregate excess demand* $z(p)$ *agrees with* $f(p)$ *on* Δ_ϵ.

Debreu's paper is hard to read, but subsequently Mantel (1976) has shown a constructive proof with homothetic utility functions.

Since the aggregate excess demand function can be virtually anything, it can be made to hit zero at arbitrary points on Δ_ϵ, so in general there can be arbitrarily many equilibria. Of course, the utility functions in these examples are quite unintuitive, and it may be possible that the equilibrium is unique in more specialized, natural economies. In fact, Debreu (1970) shows that for almost all economies there exist an odd number of equilibria $(1, 3, 5, \dots)$.

7.2 Sufficient conditions for uniqueness

Next we provide a few sufficient conditions for equilibrium uniqueness.

7.2.1 Weak Axiom of Revealed Preference for aggregate excess demand

We say that the aggregate excess demand function $z(p)$ satisfies the Weak Axiom of Revealed Preference (WARP) if for any price vectors p, p',

$$z(p) \neq z(p') \text{ and } p \cdot z(p') \leq 0 \implies p' \cdot z(p) > 0.$$

The idea of this definition is the same as that for a single consumer: if the excess demand $z(p')$ was affordable under price p but were not chosen, then $z(p)$ is revealed preferred to $z(p')$. Therefore under price p', since $z(p')$ was chosen, the excess demand $z(p)$ must not be affordable at p'.

We can show that if the aggregate excess demand satisfies WARP, then the set of equilibrium prices is convex. Therefore if the set of equilibrium prices is finite (as is generically the case by the results of Debreu (1970)), then the equilibrium must be unique. Actually WARP is stronger than necessary.

Lemma 7.2. *Suppose that the aggregate excess demand $z(p)$ satisfies WARP. Then the following condition holds:*

$$0 = z(p) \neq z(p') \implies p \cdot z(p') > 0. \tag{7.1}$$

Proof. Let p, p' be such that $0 = z(p) \neq z(p')$. Suppose on the contrary that $p \cdot z(p') \leq 0$. Then by WARP we have $p' \cdot z(p) > 0$, which is a contradiction since $z(p) = 0$. □

Proposition 7.3. *If condition (7.1) holds, then the set of equilibrium prices is convex.*

Proof. Suppose that $z(p_1) = z(p_2) = 0$. Take any $t \in [0, 1]$ and let $p = (1 - t)p_1 + tp_2$. Suppose $z(p) \neq 0$. By condition (7.1), then we have $p_1 \cdot z(p) > 0$ and $p_2 \cdot z(p) > 0$. Multiplying each inequality by $1 - t, t$ and adding up, we obtain

$$0 < (1 - t)p_1 \cdot z(p) + tp_2 \cdot z(p) = ((1 - t)p_1 + tp_2) \cdot z(p) = p \cdot z(p),$$

which contradicts the Walras law $p \cdot z(p) = 0$. Therefore $z(p) = 0$ and hence p is an equilibrium price. □

7.2.2 Gross substitute property

Proving uniqueness using WARP is not so useful because there are no known general sufficient conditions that imply WARP. Next we provide a different sufficient condition.

We say that the aggregate excess demand $z(p)$ satisfies the gross substitute (GS) property if for any l and p, p' with $p'_{l'} = p_{l'}$ for all $l' \neq l$ and $p'_l > p_l$, we have $z_{l'}(p') > z_{l'}(p)$ for all $l' \neq l$. In words, if we raise the price of good l, then the excess demand of all goods except l goes up. By the Slutsky equation, the change in the demand is the sum of the income effect and the substitution effect. Raising the price of one good makes the consumer poorer, so the demand goes down through the income effect. However, raising the price of one good makes the relative price of other goods lower, so the demand goes up through the substitution effect. Thus the gross substitute property implies that the substitution effect always dominates the income effect.

Proposition 7.4. *If the aggregate excess demand satisfies the gross substitute property, then the equilibrium is unique.*

Proof. Let p be an equilibrium price and take any q that is not collinear with p (i.e., $q \neq tp$ for any $t > 0$). By relabeling the goods if necessary, we may assume that

$$\frac{p_1}{q_1} \geq \frac{p_2}{q_2} \geq \cdots \geq \frac{p_L}{q_L}.$$

By normalizing the prices if necessary, we may assume $p_1 = q_1 = 1$. Since p, q are not collinear, at least one of the above inequalities is strict. Now starting from the price vector p, change p_l to q_l from $l = 2, \dots, L$ sequentially. At each step, since $p_l/q_l \leq 1$, we have $q_l \geq p_l$, so by the gross substitute property $z_1(p)$ goes up (or stays the same). Since there is at least one step for which $p_l/q_l < 1$, $z_1(p)$ strictly goes up at that step and thereafter. Therefore $0 = z_1(p) < z_1(q)$, so $z(q) \neq 0$. Since any q not collinear with p is not an equilibrium, the equilibrium is unique. □

Since GS holds for aggregate excess demand if each individual excess demand satisfies GS, we can provide a sufficient condition for GS. Let $x(p)$ be the demand of any agent, given prices. If $x(p)$ is differentiable, then $x(p)$ satisfies GS if $\partial x_l/\partial p_k > 0$ whenever $k \neq l$. To check this condition, we can use the implicit function theorem to compute the partial derivatives.

Proposition 7.5. *Consider an agent with initial endowment $e = (e_1, \dots, e_L)' > 0$ and utility function $U(x)$, where $x = (x_1, \dots, x_L)'$. Let $p = (p_1, \dots, p_L)' \gg 0$ be the price vector and $x = x(p)$ be the Marshallian demand function. Suppose that U is strongly monotonic ($\nabla U \gg 0$), twice continuously differentiable, and $\nabla^2 U$ is nonsingular and $p'(\nabla^2 U)^{-1} p \neq 0$ at $x(p)$. Then $x(p)$ is differentiable at p and the Jacobian is*

$$D_p x = -\lambda \left(H - \frac{1}{g' H g} H g (H g - x + e)' \right), \tag{7.2}$$

where $H = (-\nabla^2 U(x))^{-1}$, $g = \nabla U(x)$, and $\lambda > 0$ is the Lagrange multiplier on the budget constraint.

Proof. The first-order condition for utility maximization is

$$\nabla U(x) = \lambda p.$$

Since U is strongly monotonic, we have $\lambda > 0$. Define the mapping $F : \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}^L_+ \to \mathbb{R}^{L+1}$ by

$$F(x, \lambda, p) = \begin{bmatrix} \nabla U(x) - \lambda p \\ p'(e - x) \end{bmatrix}.$$

Given p, the demand x and the Lagrange multiplier λ is the solution of

$$F(x, \lambda, p) = 0.$$

Now we apply the implicit function theorem to derive the derivative of x with respect to p. For this purpose, by simple algebra we get

$$\underbrace{D_{x,\lambda}F}_{(L+1)\times(L+1)} = \begin{bmatrix} \nabla^2 U & -p \\ -p' & 0 \end{bmatrix},$$
$$\underbrace{D_p F}_{(L+1)\times L} = \begin{bmatrix} -\lambda I \\ (e-x)' \end{bmatrix}.$$

If $D_{x,\lambda}F$ is nonsingular, by the implicit function theorem we obtain

$$D_p \begin{bmatrix} x \\ \lambda \end{bmatrix} = -[D_{x,\lambda}F]^{-1} D_p F = \begin{bmatrix} \nabla^2 U & -p \\ -p' & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda I \\ (x-e)' \end{bmatrix}.$$

It is well known that for

$$M = \begin{bmatrix} A & b \\ b' & c \end{bmatrix},$$

where A is symmetric and nonsingular, we have

$$M^{-1} = \begin{bmatrix} A^{-1} + \frac{1}{k}A^{-1}bb'A^{-1} & -\frac{1}{k}A^{-1}b \\ -\frac{1}{k}b'A^{-1} & \frac{1}{k} \end{bmatrix}, \tag{7.3}$$

where $k = c - b'A^{-1}b$. Letting $A = \nabla^2 U$ (the Hessian of U), $b = -p$, and $c = 0$, it follows that $k = -p'(\nabla^2 U)^{-1}p \neq 0$ by assumption. Therefore applying formula (7.3) for $D_{x,\lambda}F = M$, we obtain

$$\begin{aligned} D_p x &= \left(A^{-1} + \frac{1}{k}A^{-1}bb'A^{-1}\right)\lambda I - \frac{1}{k}A^{-1}b(x-e)' \\ &= \lambda A^{-1} + \frac{1}{k}A^{-1}b(A^{-1}(\lambda b) - (x-e))'. \end{aligned}$$

Since $\lambda b = -\lambda p = -\nabla U$ by the first-order condition, it follows that

$$D_p x = \lambda[\nabla^2 U]^{-1} + \frac{1}{p'[\nabla^2 U]^{-1}p}[\nabla^2 U]^{-1}p(-[\nabla^2 U]^{-1}\nabla U - (x-e))'.$$

To simplify the notation, let $H = -[\nabla^2 U]^{-1}$ and $g = \nabla U = \lambda p$. Then the above expression becomes

$$D_p x = -\lambda\left(H - \frac{1}{g'Hg}Hg(Hg - x + e)'\right),$$

which is (7.2). □

Proposition 7.6 (Hens and Loeffler, 1995)**.** *Suppose that the utility function is additively separable,*

$$U(x_1, \dots, x_L) = \sum_{l=1}^{L} u_l(x_l),$$

where each von Neumann-Morgenstern utility function u_l has relative risk aversion bounded above by 1:

$$-\frac{x u_l''(x)}{u_l'(x)} \leq 1.$$

Then $\frac{\partial x_l}{\partial p_l} < 0$ *and* $\frac{\partial x_l}{\partial p_k} > 0$ *for* $l \neq k$*, so the demand satisfies the gross substitute property.*

Proof. Since $\lambda > 0$ it suffices to show that the diagonal elements of

$$D = H - \frac{1}{g'Hg} Hg(Hg - x + e)'$$

are positive and the off-diagonal elements are negative. Since U is additively separable, $H = (-\nabla^2 U)^{-1}$ is diagonal and its (l,l) element is $-\frac{1}{u_l''} > 0$. Since $g = \nabla U$, Hg is an L-vector whose l-th element is $-\frac{u_l'}{u_l''} > 0$. Therefore

$$g'Hg = -\sum_{k=1}^{L} \frac{(u_k')^2}{u_k''} > 0.$$

Combining everything, $g'Hg$ times the (l,l) element of D is

$$\begin{aligned} (g'Hg)D_{ll} &= \frac{1}{u_l''} \sum_{k=1}^{L} \frac{(u_k')^2}{u_k''} - \left(-\frac{u_l'}{u_l''}\right)^2 - \frac{u_l'}{u_l''}(x_l - e_l) \\ &= \frac{1}{u_l''} \sum_{k \neq l} \frac{(u_k')^2}{u_k''} - \frac{u_l'}{u_l''}(x_l - e_l). \end{aligned} \tag{7.4}$$

Since the utility function is strongly monotonic, we have

$$p'(x - e) = \sum_{k=1}^{L} p_k(x_k - e_k) = 0.$$

Multiplying both sides by $\lambda > 0$ and using the first-order condition $u_k' = \lambda p_k$, it follows that

$$\sum_{k=1}^{L} u_k'(x_k - e_k) = 0. \tag{7.5}$$

Therefore we obtain

$$\begin{aligned} (g'Hg)D_{ll} &= \frac{1}{u_l''} \sum_{k \neq l} \frac{(u_k')^2}{u_k''} - \frac{u_l'}{u_l''}(x_l - e_l) && (\because (7.4)) \\ &= \frac{1}{u_l''} \sum_{k \neq l} \frac{(u_k')^2}{u_k''} + \frac{1}{u_l''} \sum_{k \neq l} u_k'(x_k - e_k) && (\because (7.5)) \\ &= -\frac{1}{u_l''} \sum_{k \neq l} x_k u_k' \left(-\frac{u_k'}{x_k u_k''} - 1 + \frac{e_k}{x_k}\right). \end{aligned} \tag{7.6}$$

If the relative risk aversion of each vNM utility function is less than 1, then

$$-\frac{x_k u_k''}{u_k'} \leq 1 \iff -\frac{u_k'}{x_k u_k''} \geq 1$$

for all k, so (7.6) becomes

$$(g'Hg)D_{ll} \geq -\frac{1}{u_l''} \sum_{k \neq l} u_k' e_k > 0.$$

Since $g'Hg > 0$, we obtain $D_{ll} \geq 0$.

Similarly, the (k,l) element of $(g'Hg)D$ is

$$(g'Hg)D_{kl} = -(Hg)_k(Hg - x + e)_l = \frac{u_k'(x_k)}{u_k''(x_k)}\left(-\frac{u_l'(x_l)}{u_l''(x_l)} - x_l + e_l\right).$$

If $-\frac{x_l u_l''(x_l)}{u_l'(x_l)} \leq 1$, then $-\frac{u_l'(x_l)}{u_l''(x_l)} \geq x_l$, so since $u_k' > 0$ and $u_k'' < 0$, we have

$$(g'Hg)D_{kl} \leq \frac{u_k'(x_k)}{u_k''(x_k)} e_l < 0. \qquad \square$$

For $u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$ ($u(x) = \log x$ if $\gamma = 1$) we have $-\frac{xu''(x)}{u'(x)} = \gamma$. Therefore for CES utility functions with $\gamma \leq 1$ (in particular, Cobb-Douglas), the equilibrium is unique.

Bibliographic note

Equilibrium uniqueness is discussed in Mas-Colell et al. (1995, Chapter 17), Hens and Loeffler (1995), Kehoe (1998), Toda and Walsh (2017), and Geanakoplos and Walsh (2018b).

Exercises

7.1. This question asks you to show the uniqueness of equilibrium when agents have exponential utilities.

(i) Let $f(x) = -\frac{1}{\gamma}e^{-\gamma x}$, where $\gamma > 0$. Compute $-\frac{f''(x)}{f'(x)}$.

(ii) Consider an economy with two goods that can be consumed in any amounts (positive or negative), and suppose that an agent has endowment $e = (a, b)$ and an additively separable utility function

$$U(x, y) = u(x) + v(y),$$

where $u' > 0$, $u'' < 0$, and similarly for v. Let $p_1 = p$ and $p_2 = 1$ be prices. Show that the agent's demand (x, y) satisfies

$$u'(x) - pv'(pa + b - px) = 0.$$

(iii) Regard x, the demand for good 1, as a function of the price p. Show that

$$\frac{\partial x}{\partial p} = \frac{v'(y) + pv''(y)(a - x)}{u''(x) + p^2 v''(y)},$$

where $y = pa + b - px$.

(iv) Show that

$$\frac{\partial x}{\partial p} = -\frac{1 - p\gamma_v(y)(a - x)}{p\gamma_u(x) + p^2\gamma_v(y)},$$

where $\gamma_u(x) = -\frac{u''(x)}{u'(x)}$ and similarly for v.

(v) Suppose that there are I agents indexed by $i = 1, \dots, I$. Agent i's endowment is $e_i = (a_i, b_i)$ and the utility function is

$$U_i(x, y) = -\frac{1}{\gamma_i}\left(\alpha_i \mathrm{e}^{-\gamma_i x} + (1-\alpha_i)\mathrm{e}^{-\gamma_i y}\right),$$

where $\gamma_i > 0$ and $0 < \alpha_i < 1$. Let x_i be agent i's demand for good 1. Show that

$$\frac{\partial x_i}{\partial p} = -\frac{1}{\gamma_i p(1+p)} + \frac{1}{1+p}(a_i - x_i).$$

(vi) Let $z_1(p) = \sum_{i=1}^{I}(x_i - a_i)$ be the aggregate excess demand for good 1. Show that if p is an equilibrium price, then $z_1'(p) < 0$.

(vii) Show that the equilibrium is unique.

7.2 (Toda and Walsh, 2017)**.** This question asks you to construct an economy with multiple equilibria. Consider an economy with two agents and two goods. Suppose that the utility functions take the CES form

$$\begin{aligned} U_1(x_1, x_2) &= \frac{1}{1-\sigma}\left(\alpha^\sigma x_1^{1-\sigma} + (1-\alpha)^\sigma x_2^{1-\sigma}\right), \\ U_2(x_1, x_2) &= \frac{1}{1-\sigma}\left((1-\alpha)^\sigma x_1^{1-\sigma} + \alpha^\sigma x_2^{1-\sigma}\right), \end{aligned}$$

where $\sigma > 1$ and $0 < \alpha < 1$. The initial endowments are $e_1 = (e, 1-e)$ and $e_2 = (1-e, e)$, where $0 < e < 1$. Let $p_1 = 1$ and $p_2 = p$ be the prices, $x_{il}(p)$ be agent i's demand for good l, and

$$z_l(p) = \sum_{i=1}^{2}(x_{il}(p) - e_{il})$$

be the aggregate excess demand for good l. For notational simplicity, let $\varepsilon = 1/\sigma < 1$ be the elasticity of substitution.

(i) Compute $z_1(p)$.

(ii) Show that $z_1(0) = 0$, $z_1(1) = 0$, and $\lim_{p\to\infty} z_1(p) = \infty$.

(iii) If $z_1'(1) < 0$, show that this economy has at least three equilibria. (Hint: $p = 0$ is NOT an equilibrium. Use symmetry.)

(iv) Compute $z_1'(1)$, and show that this economy has at least three equilibria if

$$\varepsilon < 1 - \frac{1}{2}\left(\frac{e}{\alpha} + \frac{1-e}{1-\alpha}\right).$$

(v) Show that if $\sigma > 2$, then we can construct an economy with at least three equilibria.

(vi) Let $\sigma = 3$, $\alpha = 1/7$, and $e = 1/49$. Write a computer program that computes the excess demand $z_1(p)$ and plot the results over $10^{-1} \le p \le 10^1$ in a semi log scale (make sure to plot the horizontal axis so that we can see where the excess demand is zero). Show that this economy has three equilibria, $p = 1/8, 1, 8$.

7.3. If you give a charity, assuming that the good is valuable and there is no externality (so you care only about your own consumption and the recipient cares only about his or her own consumption), you would imagine that you will be worse off and the recipient better off. This question asks you to show that this intuition is wrong: by giving a valuable good to another person, you might actually become better off and hurt the other, so a charity may not benefit the recipient but instead the donor!

Consider the same economy as above, and assume that

$$0 < \frac{1}{\sigma} < 1 - \frac{1}{2}\left(\frac{e}{\alpha} + \frac{1-e}{1-\alpha}\right),$$

so there are at least three equilibria. Suppose that agent 1 gives $0 < t < 1 - e$ units of good 2 to agent 2, so the endowments are $e_1 = (e, 1 - e - t)$ and $e_2 = (1 - e, e + t)$ after the transfer.

(i) Consider the equilibrium price $p = 1$ when there is no transfer ($t = 0$). Let $p(t)$ be the new equilibrium price closest to 1 when $|t|$ is small. Compute $p'(0)$.

(ii) Consider an agent with CES utility

$$U(x_1, x_2) = \frac{1}{1-\sigma}\left(\alpha_1^\sigma x_1^{1-\sigma} + \alpha_2^\sigma x_2^{1-\sigma}\right)$$

and wealth $w > 0$. Using the result of Problem Set 1, compute the maximum utility of the agent when the price vector is $p = (p_1, p_2)$ and the wealth is $w > 0$.

(iii) Let $V_i(p, t) = \frac{1}{1-\sigma}\log((1-\sigma)U_i(x_{i1}, x_{i2}))$ (which is a monotonic transformation of $U(x_1, x_2)$), where the utility is evaluated at the demand when the price is $(1, p)$ and the transfer is t. Evaluate the change in welfare $\frac{\mathrm{d}}{\mathrm{d}t}V_i(p(t), t)$ at $t = 0$.

(iv) Show that if $\frac{1}{\sigma} < 1 - \frac{1}{2}\left(\frac{e}{\alpha} + \frac{1-e}{1-\alpha}\right)$ and the current equilibrium price is $(1, 1)$, then a small gift of good 2 from agent 1 to 2 benefits agent 1 and hurts agent 2. Using a computer, compute the new equilibrium price and welfare when $\sigma = 3$, $\alpha = 1/7$, $e = 1/49$, and $t = 10^{-3}$, and verify the above claim.

7.4 (Geanakoplos and Walsh, 2018a)**.** Consider the following general equilibrium model. There are three time periods indexed by $t = 0, 1, 2$. There is a continuum of ex ante identical agents, where the population is normalized to 1. At $t = 0$, agents are endowed with $e > 0$ units of consumption good. At $t = 0$, agents can invest goods in two technologies. One unit of investment in technology 1 yields 1 unit of good at $t = 1$. One unit of investment in technology 2 yields $R > 0$ units of good at $t = 2$. Agents get utility only from consumption at $t = 1, 2$. At the beginning of $t = 1$, agents get "liquidity shocks", and with probability $\pi_i > 0$, their utility function becomes

$$U_i(x_1, x_2) = (1 - \beta_i)\log x_1 + \beta_i \log x_2,$$

where $\beta_i \in (0,1)$ is the discount factor of type i and $\sum_{i=1}^I \pi_i = 1$. Without loss of generality, assume

$$\beta_1 < \cdots < \beta_I,$$

so a type with a smaller index is more impatient. Suppose that the ex ante utility is

$$U((x_{i1}, x_{is})_i) = \sum_{i=1}^I \pi_i \alpha_i U_i(x_{i1}, x_{i2}),$$

where $\alpha_i > 0$ is the weight on type i such that

$$\alpha_1 > \cdots > \alpha_I,$$

so agents care about emergencies in the sense that they put more utility weight on the impatient type. Note that we assume the law of large numbers, so at $t = 1$, exactly fraction $\pi_i > 0$ of agents are of type i. After observing their patience type at $t = 1$, agents can trade consumption for $t = 1, 2$ at a competitive (Arrow-Debreu) market.

(i) In general, let f, g be strictly increasing functions and X be a random variable. Prove the Chebyshev inequality

$$\mathrm{E}[f(X)g(X)] \geq \mathrm{E}[f(X)]\,\mathrm{E}[g(X)],$$

with equality if and only if X is constant almost surely.

(Hint: let X' be an i.i.d. copy of X and consider the expectation of the quantity $(f(X) - f(X'))(g(X) - g(X')) \geq 0$.)

(ii) Noting that agents are ex ante identical, at $t = 0$ they will all make the same investment decision. let $x \in (0, e)$ be the amount of investment in technology 1 and let $(e_1, e_2) = (x, R(e - x))$ be the vector of $t = 1, 2$ endowments conditional on x. Let $(p_1, p_2) = (1, p)$ be the price of consumption at $t = 1, 2$. Compute type i's demand for the $t = 1, 2$ goods using p, e_1, e_2.

(iii) For notational simplicity, let $\bar{\beta} = \sum_{i=1}^I \pi_i \beta_i$ be the average discount factor. Conditional on x, compute the equilibrium price p.

(iv) Let $V(x, p) = \max U((x_{i1}, x_{i2})_i)$ be the agents' maximized utility conditional on short-term investment x and price p. Noting that agents choose x optimally given p, compute the equilibrium short-term investment x^*.

(v) Suppose that the government can force the agents to choose a particular x, without interfering in the subsequent consumption markets at $t = 1, 2$. Let $p(x)$ be the price of $t = 2$ consumption conditional on x derived above. Prove that

$$\left.\frac{\mathrm{d}}{\mathrm{d}x} V(x, p(x))\right|_{x = x^*} > 0,$$

so welfare locally increases if the government forces the agents to invest more in the short-term investment technology.

Chapter 8

Computation of equilibrium

Computing the equilibrium is usually difficult (at least by hand) because for a given price we need to compute the demand of each agent (solving as many constrained optimization problems as the number of agents) and then find a price that clears all markets simultaneously (solving as many nonlinear equations as the number of goods). We have already seen that when utilities are quasi-linear, the computation of the equilibrium is straightforward because all we need is to maximize the sum of the nonlinear part of the utilities subject to the feasibility constraint, and the Lagrange multiplier will give us the price vector.

This chapter introduces other models for which the computation of equilibrium is relatively easy, which is useful for applied works.

8.1 Homothetic preferences

A preference relation $\succsim$ is said to be *homothetic* if the preference ordering is unchanged by scaling up or down commodity bundles, that is, for all $\lambda > 0$ we have $x \succsim y \implies \lambda x \succsim \lambda y$. If the preference relation has a utility function representation u, then it means that $u(x) \geq u(y) \implies u(\lambda x) \geq u(\lambda y)$. If u is homogeneous of degree 1, so $u(\lambda x) = \lambda u(x)$ for all $\lambda > 0$, then u is clearly homothetic. The following proposition shows that the converse is true if the preference is weakly monotonic and continuous.

Proposition 8.1. *Let $u : \mathbb{R}^L_+ \to \mathbb{R}$ be a weakly monotonic homothetic utility function that is continuous on $\mathbb{R}^L_{++}$. Then there exists a strictly increasing function $f : \mathbb{R}_{++} \to \mathbb{R}$ such that $v(x) := f^{-1}(u(x))$ is homogeneous of degree 1, i.e., for all $\lambda > 0$ and $x \in \mathbb{R}^L_{++}$ we have $v(\lambda x) = \lambda v(x)$.*

Proof. Take any $a \in \mathbb{R}^L_{++}$ (say $a = (1, \dots, 1)'$) and define $f(t) = u(ta)$ for $t > 0$. Since u is weakly monotonic, f is strictly increasing. Since u is continuous, so is f. Take any $x \gg 0$. Since $a \gg 0$, we can take $0 < t_1 < t_2$ such that $0 \ll t_1 a \ll x \ll t_2 a$. Applying u to both sides, since u is weakly monotonic, we get

$$f(t_1) = u(t_1 a) < u(x) < u(t_2 a) = f(t_2).$$

Since f is continuous and strictly increasing, by the intermediate value theorem there exists a unique t such that $u(x) = f(t) = u(ta)$. Define $v(x) = t =$

$f^{-1}(u(x))$. Since u is homothetic, we have $u(\lambda x) = u(\lambda ta) = f(\lambda t)$. Applying f^{-1} to both sides, it follows that

$$v(\lambda x) = f^{-1}(u(\lambda x)) = f^{-1}(f(\lambda t)) = \lambda t = \lambda f^{-1}(u(x)) = \lambda v(x). \qquad \square$$

Example 8.1. The Cobb-Douglas utility function

$$u(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2$$

is homothetic. In fact, letting $f(t) = \log t$,

$$v(x_1, x_2) := f^{-1}(u(x_1, x_2)) = \exp(u(x_1, x_2)) = x_1^\alpha x_2^{1-\alpha}$$

is homogeneous of degree 1.

Example 8.2. The CES utility function

$$u(x_1, x_2) = \frac{1}{1-\gamma}(\alpha_1 x_1^{1-\gamma} + \alpha_2 x_2^{1-\gamma})$$

is homothetic. In fact, letting $f(t) = \frac{1}{1-\gamma} t^{1-\gamma}$,

$$v(x_1, x_2) = f^{-1}(u(x_1, x_2)) = ((1-\gamma)u(x_1, x_2))^{\frac{1}{1-\gamma}} = (\alpha_1 x_1^{1-\gamma} + \alpha_2 x_2^{1-\gamma})^{\frac{1}{1-\gamma}}$$

is homogeneous of degree 1.

Example 8.3. The Leontief utility function

$$u(x_1, x_2) = \min\left\{\frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2}\right\}$$

is homogeneous of degree 1 and hence homothetic.

A remarkable property of functions that are homogeneous of degree 1 is that quasi-convexity (concavity) implies convexity (concavity). Let X be a vector space over $\mathbb{R}$. Recall that a set $C \subset X$ is a *cone* if for any $x \in C$ and $\alpha \geq 0$, we have $\alpha x \in C$. The following proposition is slightly stronger than Berge (1963, p. 208, Theorem 3).

Proposition 8.2. *Let X be a vector space and $C \subset X$ a convex cone. Suppose that $f : C \to \mathbb{R} \cup \{\pm\infty\}$ is* (i) *homogeneous of degree 1,* (ii) *quasi-convex (concave), and* (iii) *either $f(x) > 0$ for all $x \in C \setminus \{0\}$ or $f(x) < 0$ for all $x \in C \setminus \{0\}$. Then f is convex (concave).*

Proof. It suffices to show convexity since concavity follows from replacing f by $-f$. Let us show

$$f((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)f(x_1) + \alpha f(x_2). \tag{8.1}$$

Since f is homogeneous of degree 1, we have $f(0) = f(0x) = 0f(x) = 0$.

If $x_1 = 0$, using homogeneity we obtain

$$\begin{aligned} f((1-\alpha)x_1 + \alpha x_2) &= f(\alpha x_2) = \alpha f(x_2) \\ &= (1-\alpha)0 + \alpha f(x_2) = (1-\alpha)f(x_1) + \alpha f(x_2), \end{aligned}$$

so (8.1) holds. Similarly, (8.1) holds if $x_2 = 0$.

Suppose $x_1, x_2 \neq 0$. By assumption, $f(x_1), f(x_2)$ are both nonzero and have the same sign. Since (8.1) is trivial if $\alpha = 0, 1$, we may assume $\alpha \in (0, 1)$. Then $(1-\alpha)f(x_1)$ and $\alpha f(x_2)$ are also both nonzero and have the same sign. Take $k > 0$ such that $k(1-\alpha)f(x_1) = \alpha f(x_2)$. Let $\bar{x} = (1-\alpha)x_1 + \alpha x_2$. Using the homogeneity and quasi-convexity of f, we obtain

$$\begin{aligned}\frac{k}{1+k}f(\bar{x}) &= f\left(\frac{k}{1+k}\bar{x}\right) = f\left(\frac{1}{1+k}k(1-\alpha)x_1 + \frac{k}{1+k}\alpha x_2\right)\\ &\leq \max\left\{f(k(1-\alpha)x_1), f(\alpha x_2)\right\} = \max\left\{k(1-\alpha)f(x_1), \alpha f(x_2)\right\}.\end{aligned}$$

Since by construction $k(1-\alpha)f(x_1) = \alpha f(x_2)$, the last expression is also equal to

$$\frac{1}{1+k}k(1-\alpha)f(x_1) + \frac{k}{1+k}\alpha f(x_2) = \frac{k}{1+k}((1-\alpha)f(x_1) + \alpha f(x_2)).$$

Therefore

$$\frac{k}{1+k}f(\bar{x}) \leq \frac{k}{1+k}((1-\alpha)f(x_1) + \alpha f(x_2)),$$

and dividing both sides by $\frac{k}{1+k} > 0$, we obtain (8.1). □

8.2 Identical homothetic preferences with arbitrary endowments

If agents have identical convex homothetic preferences, then the equilibrium price is the same as in an economy consisting of a single agent (representative agent) with the same preferences, endowed with the aggregate endowment. Individual consumption is then the aggregate endowment scaled by individual wealth.

Proposition 8.3. *Consider an economy $\mathcal{E} = \{I, (e_i), (u_i)\}$ with identical smooth homothetic preferences, so $u_i(x) = u(x) = f(v(x))$ for all i, where $f' > 0$ and v is C^1 and homogeneous of degree 1. Let $e = \sum_{i=1}^I e_i \gg 0$ be the aggregate endowment. If u is weakly monotonic, quasi-concave, and $\nabla u(e) \neq 0$, then $p = \nabla u(e)$ and the allocation $x_i = \frac{p \cdot e_i}{p \cdot e}e$ constitute an equilibrium. If u is strictly quasi-concave, then the equilibrium is unique.*

Proof. Since $\nabla u(x) = f'(v(x))\nabla v(x)$ by the chain rule, if $p = \nabla u(e)$ is a price vector, so is $q = tp = \nabla v(e)$ by setting $t = \frac{1}{f'(u(e))} > 0$. Therefore without loss of generality we may assume that u itself is homogeneous of degree 1.

Differentiating $u(\lambda x) = \lambda u(x)$ with respect to x, we get

$$\nabla u(\lambda x)\lambda = \lambda \nabla u(x) \iff \nabla u(\lambda x) = \nabla u(x),$$

so $\nabla u(x)$ is homogeneous of degree 0. Let $p = \nabla u(e) \neq 0$. Since u is weakly monotonic, we have $p > 0$. Therefore $p \cdot e > 0$. Consider the allocation $x_i = \frac{p \cdot e_i}{p \cdot e}e$. Since

$$\sum_{i=1}^I x_i = \sum_{i=1}^I \frac{p \cdot e_i}{p \cdot e}e = \frac{p \cdot e}{p \cdot e}e = e,$$

(x_i) is feasible. Let

$$L(x, \lambda_i) = u(x) + \lambda_i(p \cdot e_i - p \cdot x)$$

be the Lagrangian of the utility maximization problem of agent i. The first-order condition is $\nabla u(x) = \lambda_i p$. By setting $p = \nabla u(e)$ and $\lambda_i = 1$, since $\nabla u(x)$ is homogeneous of degree 0, the first-order condition holds at $x_i = \frac{p \cdot e_i}{p \cdot e} e$. Hence by the Karush-Kuhn-Tucker theorem for quasi-concave functions, x_i solves the utility maximization problem. Therefore $\{p, (x_i)\}$ is an equilibrium.

If u is strictly quasi-concave, then the solution to the utility maximization problem is unique. Since preferences are identical and homothetic, using market clearing all demands must be a positive multiple of the aggregate endowment. By the first-order condition, the price vector must be a positive multiple of $p = \nabla u(e)$. Since weakly monotonic preferences are locally non-satiated, individual demand must then be $x_i = \frac{p \cdot e_i}{p \cdot e} e$. Therefore the equilibrium is unique. $\square$

The above theorem is sometimes called "Gorman aggregation" after Gorman (1953). When agents have identical homothetic preferences, we can treat the economy as if there is a single (representative) agent with the same preferences consuming the aggregate endowment.[1]

8.3 Arbitrary homothetic preferences with collinear endowments

Next we consider arbitrary homothetic preferences but assume that endowments are collinear, so $e_i = w_i e$ with $w_i > 0$ and $\sum_{i=1}^I w_i = 1$, where $e \in \mathbb{R}^L_{++}$ is the aggregate endowment and $w_i > 0$ is the wealth share of agent i.

Proposition 8.4. *Let everything be as above. Let $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ be the utility function of agent i. Suppose that u_i is continuous, quasi-concave, homogeneous of degree 1, and satisfies $\nabla u_i \gg 0$ on $\mathbb{R}^L_{++}$. Let (x_i) be an interior feasible allocation (so $x_i \gg 0$ for all i and $\sum_{i=1}^I x_i \leq e$). Then (x_i) is an equilibrium allocation if and only if it solves the planner's problem*

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^I w_i \log u_i(y_i) \\ &\text{subject to} && \sum_{i=1}^I y_i \leq e, \end{aligned}$$

in which case the price vector is proportional to the Lagrange multiplier. If each u_i is strictly quasi-concave, then the equilibrium is unique.

Proof. First we need to show that $u_i(x) \geq 0$ so that $\log u_i(x)$ makes sense. Since u_i is homogeneous of degree 1, we have $u_i(\lambda x) = \lambda u_i(x)$. Letting $\lambda \to 0$, we get $u_i(0) = 0$. Since u_i is weakly monotonic, for any $x \gg 0$ we have

[1] Many papers in macroeconomics and finance make such assumptions, although it is quite unrealistic.

$u_i(x) > u_i(0) = 0$, so $\log u_i(x)$ is well-defined. By Proposition 8.2, $u_i(x)$ is concave, and so is $\log u_i(x)$ because $\log(\cdot)$ is increasing and concave. Let

$$L(y_1, \dots, y_I, p) = \sum_{i=1}^{I} w_i \log u_i(y_i) + p \cdot \left(e - \sum_{i=1}^{I} y_i \right)$$

be the Lagrangian of the planner's problem, where $p \in \mathbb{R}^L_+$ is the vector of Lagrange multipliers. If (x_i) is a solution, by the KKT theorem we have $w_i \frac{\nabla u_i(x_i)}{u_i(x_i)} = p$ for all i. Since $\nabla u_i(x_i) > 0$ and $u_i(x_i) > 0$, we have $p > 0$.

To show that $\{p, (x_i)\}$ constitute an equilibrium, it suffices to show agent optimization and market clearing. Since (x_i) solves the planner's problem, it is clearly feasible. Let $\lambda_i = \frac{u_i(x_i)}{w_i} > 0$. Let us show that x_i solves the utility maximization problem of agent i with Lagrange multiplier λ_i. To show that x_i satisfies the budget constraint, take any $\lambda > 0$ and $x \gg 0$. Differentiating both sides of $u(\lambda x) = \lambda u(x)$ with respect to λ and setting $\lambda = 1$, we get $\nabla u(x) \cdot x = u(x)$. Therefore

$$p \cdot x_i = w_i \frac{\nabla u_i(x_i)}{u_i(x_i)} \cdot x_i = w_i.$$

Adding across i and using complementary slackness condition, we get

$$1 = \sum_{i=1}^{I} w_i = p \cdot \sum_{i=1}^{I} x_i = p \cdot e,$$

so

$$p \cdot x_i = w_i = w_i p \cdot e = p \cdot (w_i e) = p \cdot e_i.$$

Therefore x_i satisfies the budget constraint. Since $\nabla u_i(x_i) = \lambda_i p > 0$, by the KKT theorem x_i solves the utility maximization problem.

Conversely, suppose that $\{p, (x_i)\}$ is an equilibrium. By normalizing the price if necessary, we may assume $p \cdot e = 1$. Then $\nabla u_i(x_i) = \lambda_i p$ for some $\lambda_i > 0$, and hence

$$u_i(x_i) = \nabla u_i(x_i) \cdot x_i = \lambda_i p \cdot x_i = \lambda_i p \cdot e_i = \lambda_i w_i \iff \lambda_i = \frac{u_i(x_i)}{w_i}.$$

Substituting into the first-order condition for utility maximization, we get

$$\nabla u_i(x_i) = \frac{u_i(x_i)}{w_i} p \iff w_i \frac{\nabla u_i(x_i)}{u_i(x_i)} = p.$$

But this equation shows that the first-order condition of the planner's problem holds with Lagrange multiplier p. Since each $\log u_i$ is concave, the first-order condition is sufficient, so (x_i) solves the planner's problem. If $\log u_i$ are strictly concave, then the planner's problem has a unique solution, so the equilibrium is unique. □

Chipman (1974) shows that aggregation is possible with arbitrary homothetic preferences and collinear endowments, although with a different argument. Proposition 8.4 is essentially due to Chipman and Moore (1979).

8.4 Aggregation with HARA preferences

So far we assumed that preferences are homothetic. What if preferences are non-homothetic? Under certain conditions, the economy still admits a representative agent and therefore the computation of equilibrium is straightforward.

8.4.1 Risk aversion

To deal with this case, it would be more convenient to interpret goods as the same physical good but available in different states, so we label goods by $s = 1, 2, \dots, S$ instead of $l = 1, 2, \dots, L$. Let $\pi_s > 0$ be the probability of state s and consider an agent (investor) with expected utility

$$\mathrm{E}[u(x)] = \sum_{s=1}^{S} \pi_s u(x_s),$$

where $x = (x_1, \dots, x_S)$ is the consumption bundle and $u' > 0$ (increasing) and $u'' < 0$ (concave).

Suppose that the investor has initial wealth w. Let ϵ be a small gamble, so $\mathrm{E}[\epsilon] = 0$. Consider the following two options.

(i) The gamble enters the investor's wealth additively, so his expected utility is $\mathrm{E}[u(w + \epsilon)]$.

(ii) The investor does not hold the gamble but gives up $a > 0$, so his utility is $u(w - a)$.

When is the investor indifferent between these two options? Of course, the answer is when a satisfies

$$\mathrm{E}[u(w + \epsilon)] = u(w - a).$$

Noting that ϵ is a small gamble, we can Taylor-expand the left-hand side to the second order:

$$\mathrm{E}[u(w + \epsilon)] \approx \mathrm{E}\left[u(w) + u'(w)\epsilon + \frac{1}{2}u''(w)\epsilon^2\right] = u(w) + \frac{1}{2}u''(w)\operatorname{Var}[\epsilon],$$

where I used $\mathrm{E}[\epsilon] = 0$ and $\mathrm{E}[\epsilon^2] = \operatorname{Var}[\epsilon]$. (The reason why I expanded to the second order is because the first order term is zero.) Similarly, Taylor-expanding the right-hand side to the first order, we get

$$u(w - a) \approx u(w) - u'(w)a.$$

Putting everything together and replacing $\approx$ by $=$, we get

$$\begin{aligned} \mathrm{E}[u(w + \epsilon)] = u(w - a) &\iff u(w) + \frac{1}{2}u''(w)\operatorname{Var}[\epsilon] = u(w) - u'(w)a \\ &\iff a = -\frac{u''(w)}{u'(w)}\frac{\operatorname{Var}[\epsilon]}{2}. \end{aligned}$$

The last equation shows that the investor should give up an amount proportional to the variance in order to avoid the risk. The term

$$\mathrm{ARA}(w) = -\frac{u''(w)}{u'(w)} > 0$$

is called the *absolute risk aversion coefficient* at wealth w. ARA is sometimes referred to as the Arrow-Pratt measure of absolute risk aversion, after Arrow (1964) and Pratt (1964). The reciprocal of the absolute risk aversion $-\frac{u'(w)}{u''(w)}$ is called the *absolute risk tolerance*.

8.4.2 HARA utility function

In applied work, it is often convenient when the risk tolerance is linear in wealth, so

$$-\frac{u'(w)}{u''(w)} = aw + b$$

for some constants a, b. (Here we consider only the wealth level for which $aw + b > 0$.) Such a utility function is called *linear risk tolerance*, or LRT for short. Since the absolute risk aversion is

$$-\frac{u''(w)}{u'(w)} = \frac{1}{aw + b},$$

whose graph is a hyperbola, it is also called *hyperbolic absolute risk aversion*, or HARA for short. We can easily characterize all HARA utilities by solving a differential equation. Assume $a \neq 0$. Integrating once, we get

$$\log u'(w) = -\frac{1}{a}\log(aw + b) + \text{constant}.$$

Taking the exponential, we get

$$u'(w) = C(aw + b)^{-\frac{1}{a}}$$

for some $C > 0$. Assuming $a \neq 1$ and integrating once again, we get

$$u(w) = \frac{C}{a - 1}(aw + b)^{1-\frac{1}{a}} + D,$$

where D is some constant. Since C, D merely defines an affine transformation, they do not affect the ordering of expected utility. Therefore without loss of generality we may assume $C = 1$ and $D = 0$. We can also consider the cases $a = 0, 1$ separately, and the result is

$$u(w) = \begin{cases} \frac{1}{a-1}(aw + b)^{1-1/a}, & (a \neq 0, 1) \\ -b\mathrm{e}^{-w/b}, & (a = 0) \\ \log(w + b). & (a = 1) \end{cases}$$

Example 8.4. If we set $a = \frac{1}{\gamma}$ and $b = 0$, then we get

$$u(w) = \frac{1}{\frac{1}{\gamma} - 1}(w/\gamma)^{1-\gamma} = \frac{\gamma^\gamma}{1 - \gamma}w^{1-\gamma},$$

which is the same as the CES utility function.

Example 8.5. If we set $a = 1$ and $b = 0$, then we get $u(w) = \log w$, which is the same as Cobb-Douglas. If $a = 1$ but $b < 0$, then the utility function is sometimes called Stone-Geary.[2]

Example 8.6. If we set $a = -1$ and $b > 0$, then we get $u(w) = -\frac{1}{2}(b - w)^2$, the quadratic utility.

[2] https://en.wikipedia.org/wiki/Stone-Geary_utility_function

8.4.3 Aggregation

When agents have HARA preferences with identical parameter a, then the economy admits a representative agent.

Proposition 8.5. *Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with HARA utilities. Let (a_i, b_i) be the HARA parameter of agent i, where $a_i = a$ is common across agents. Then the economy admits a HARA representative agent with parameter (a, b), where $b = \sum_{i=1}^I b_i$. Letting $e = \sum_{i=1}^I e_i$ be the aggregate endowment, the price is given by*

$$p_s = \begin{cases} \pi_s (ae_s + b)^{-\frac{1}{a}}, & (a \neq 0) \\ \pi_s \mathrm{e}^{-\frac{e_s}{b}}. & (a = 0) \end{cases}$$

The allocation satisfies

$$\begin{cases} ax_{is} + b_i = \lambda_i^{-a}(ae_s + b), & (a \neq 0) \\ \frac{x_{is}}{b_i} = -\log \lambda_i + \frac{e_s}{b}, & (a = 0) \end{cases}$$

where $\lambda_i > 0$ is agent i's Lagrange multiplier that is determined from the budget constraint.

Proof. Assume $a \neq 0$. (The case $a = 0$ is left as an exercise.) Let $\{p, (x_i)\}$ be an equilibrium. By the first-order condition of agent i with respect to good s, we obtain

$$\pi_s (ax_{is} + b_i)^{-\frac{1}{a}} = \lambda_i p_s,$$

where $\lambda_i > 0$ is the Lagrange multiplier. Using this equation for general s and $s = 1$ to eliminate λ_i, we obtain

$$\begin{aligned} \frac{\pi_s}{p_s}(ax_{is} + b_i)^{-\frac{1}{a}} &= \frac{\pi_1}{p_1}(ax_{i1} + b_i)^{-\frac{1}{a}} \\ \iff \left(\frac{\pi_s}{p_s}\right)^{-a} (ax_{is} + b_i) &= \left(\frac{\pi_1}{p_1}\right)^{-a} (ax_{i1} + b_i). \end{aligned}$$

Adding across agents, using market clearing, and letting $e_s = \sum_{i=1}^I e_{is}$ be the aggregate endowment in state s, we obtain

$$\left(\frac{\pi_s}{p_s}\right)^{-a} (ae_s + b) = \left(\frac{\pi_1}{p_1}\right)^{-a} (ae_1 + b) \iff \frac{\pi_s}{p_s}(ae_s + b)^{-\frac{1}{a}} = \frac{\pi_1}{p_1}(ae_1 + b)^{-\frac{1}{a}}.$$

Therefore the quantity $\lambda := \frac{\pi_s}{p_s}(ae_s + b)^{-\frac{1}{a}}$ does not depend on s. Since price levels do not matter for equilibrium, without loss of generality we may assume $\lambda = 1$ and hence

$$\pi_s (ae_s + b)^{-\frac{1}{a}} = p_s.$$

However, this is precisely the first-order condition of the representative agent with HARA utility (a, b) consuming the aggregate endowment, where the Lagrange multiplier is $\lambda = 1$. Substituting this price into agent i's first-order condition, we obtain

$$\pi_s (ax_{is} + b_i)^{-\frac{1}{a}} = \lambda_i \pi_s (ae_s + b)^{-\frac{1}{a}} \iff ax_{is} + b_i = \lambda_i^{-a}(ae_s + b). \qquad \square$$

Chapter 9

International trade

9.1 Numerical example of Ricardo's model

Consider an economy with two countries, A, B. Think of country A as developed and B as developing. Country A and B have labor endowment $e_A = 1$ and $e_B = 2$, respectively. Each country produces two identical consumption goods, $1, 2$. Technology is linear: if country i employs labor e in technology l, it produces $a_{il}e$ units of good l, where $a_{il} > 0$ is productivity. Assume

$$(a_{A1}, a_{A2}, a_{B1}, a_{B2}) = (10, 5, 4, 1),$$

so country A is way more productive. (Think of good 1 as agricultural product and good 2 as high-tech manufacturing.) Assume each country has utility function

$$u(x_1, x_2) = x_1 x_2.$$

Note that this utility function is the same as the Cobb-Douglas utility function

$$\frac{1}{2}\log x_1 + \frac{1}{2}\log x_2.$$

First let us compute the equilibrium when there is no international trade (this is called *autarky*).

Country A Let the price be $p_1 = 1$, $p_2 = p$, and the wage be w. Since labor supply is $e_A = 1$, the labor income (also GDP) is $we_A = w$. Using the Cobb-Douglas formula, the demand of each good is

$$(x_1, x_2) = \left(\frac{w}{2}, \frac{w}{2p}\right).$$

Assume that the firm producing good l hires labor e_l. Noting that the output of good l is $y_l = a_{Al}e_l$, firm l's profit is

$$p_l a_{Al} e_l - we_l = (p_l a_{Al} - w)e_l.$$

The firm maximizes profit. Since the profit function is linear in labor input e_l, in order for a maximum to exist the slope must be zero. Therefore $w = p_l a_{Al}$.

It follows that $w = p_1 a_{A1} = 10$ and $p = p_2 = w/a_{A2} = 10/5 = 2$. Substituting into the demand above, we get

$$(x_1, x_2) = \left(\frac{w}{2}, \frac{w}{2p}\right) = \left(5, \frac{5}{2}\right).$$

Utility is

$$u(x_1, x_2) = x_1 x_2 = \frac{25}{2}.$$

We have not used market clearing yet. In order to consume 5 units of good 1 using the technology $y_1 = 10e_1$, it has to be $e_1 = 5/10 = \frac{1}{2}$. Therefore $e_2 = e_A - e_1 = \frac{1}{2}$, so labor is equally divided between the two industries. Note that $x_2 = 5/2 = a_{A2}e_2$, as it should be.

Country B This is left as an exercise. The answer should be $w = 4$, $p_2 = 4$, $(x_1, x_2) = (4, 1)$ with utility level 4, and labor allocation $(e_1, e_2) = (1, 1)$.

Free trade Assume now that there is international trade, so labor cannot move across countries but goods are free to move. In reality people often misunderstand the benefit of free trade. For example, in the above example the wage in country A is two times that in country B, so residents of country A might fear that domestic jobs will be lost by opening up to trade. On the other hand, since the industries of country B are much less productive, residents of country B might fear that domestic industries will decline by opening up to trade. Both opinions are incorrect. To see this, let us compute the international trade equilibrium.

Computing the equilibrium for Ricardo's model of international trade requires some guesswork. First, note that in autarky the price of good 2 is $p_2 = 2$ in country A and $p_2 = 4$ in country B. Since good 2 is much more expensive in country B, after free trade it is reasonable to assume that country B will import good 2 and export good 1. The opposite is true for country A. Let $p_1 = 1$ and $p_2 = p$ be the world price in free trade. Since country A produces good 2 (since it is exporting), by firm's profit maximization we have

$$p_2 a_{A2} - w_A = 0 \iff w_A = 5p.$$

Since labor endowment is $e_A = 1$, total labor income (GDP) is $w_A e_A = 5p$. Using the Cobb-Douglas formula, the demand of country A is

$$(x_{A1}, x_{A2}) = \left(\frac{5p}{2}, \frac{5}{2}\right).$$

We do the same for country B. Since country B produces good 1, by firm's profit maximization we have

$$p_1 a_{B1} - w_B = 0 \iff w_B = 4.$$

Since labor endowment is $e_B = 2$, total labor income is $w_B e_B = 8$, and the demand of country B is

$$(x_{B1}, x_{B2}) = \left(4, \frac{4}{p}\right).$$

Now we compute the equilibrium, which is where we have to guess (or compare two cases). Suppose that country B also produces good 2. Then by the same argument as in the autarky case, we have $p = 4$, so the aggregate demand of good 1 is

$$\frac{5p}{2} + 4 = 10 + 4 = 14.$$

Since both countries cannot be producing both goods (otherwise the price must be 2 and 4 at the same time, which is impossible), it must be the case that country B is producing the entire amount of good 1 in the world. Since the labor endowment of country B is $e_B = 2$, the maximum amount of good 1 country B can produce is

$$a_{B1}e_B = 4 \times 2 = 8 < 14,$$

so the market cannot clear.

Now suppose that country A also produces good 1. Then by the same argument as in the autarky case, we have $p = 2$, so the aggregate demand of good 2 is

$$\frac{5}{2} + \frac{4}{p} = \frac{5}{2} + 2 = \frac{9}{2}.$$

Since both countries cannot be producing both goods (otherwise the price must be 2 and 4 at the same time, which is impossible), it must be the case that country A is producing the entire amount of good 2 in the world. Since the labor endowment of country A is $e_A = 1$, the necessary labor input for producing good 2 to clear market is

$$\frac{9}{2} = a_{A2}e_{A2} \iff e_{A2} = \frac{9}{10},$$

which is feasible.

Therefore the equilibrium price is $(p_1, p_2) = (1, 2)$, consumption is $(x_{A1}, x_{A2}) = (5, 5/2)$ and $(x_{B1}, x_{B2}) = (4, 2)$, and labor allocation is $(e_{A1}, e_{A2}) = (1/10, 9/10)$ and $(e_{B1}, e_{B2}) = (2, 0)$. Note that the consumption of country A is the same in autarky and in free trade equilibrium, but the consumption of country B has changed from $(4, 1)$ to $(4, 2)$. Therefore free trade is Pareto improving, and the more inefficient country B gained from trade.

9.2 General case and comparative advantage

Now consider the more general case where there are two countries $i = A, B$ and L goods $l = 1, \dots, L$. Country i has labor endowment e_i and can produce good l according to the technology $y = a_{il}e$, where e is labor input and y is output of good l. We can do as follows to solve for the free trade equilibrium. First, for each good compute the ratio of productivities (called *comparative advantage* of country A over B for good l) a_{Al}/a_{Bl} and relabel the goods so that

$$\frac{a_{A1}}{a_{B1}} > \dots > \frac{a_{AL}}{a_{BL}}.$$

This inequality shows that country A is relatively most productive in producing good 1, and relatively least productive in producing good L.

Let us show that in equilibrium countries specialize in the production of goods that they have comparative advantage. In particular, there exists a good

l^* such that all goods $l < l^*$ are produced by country A, and all goods $l > l^*$ are produced by country B. (The good l^* may or may not be produced by both countries.) Here is the proof. Let

$$l^* = \max_l \{\text{good } l \text{ is produced by country } A\}.$$

By definition, all goods $l > l^*$ are produced by country B. To show that all goods $l < l^*$ are produced by country A, suppose on the contrary that there is a good $l < l^*$ produced by country B. Then by the zero profit condition we have

$$p_l a_{Bl} = w_B.$$

Good l^* may or may not be produced by country B, but in any case B cannot be making positive profit. Therefore

$$p_{l^*} a_{Bl^*} \le w_B.$$

By definition, good l^* is produced by country A. By the zero profit condition, we have

$$p_{l^*} a_{Al^*} = w_A.$$

Good l may or may not be produced by country A, but in any case A cannot be making positive profit. Therefore

$$p_l a_{Al} \le w_A.$$

Dividing the fourth equation by the first, we obtain

$$\frac{a_{Al}}{a_{Bl}} \le \frac{w_A}{w_B}.$$

Dividing the third equation by the second, we obtain

$$\frac{a_{Al^*}}{a_{Bl^*}} \ge \frac{w_A}{w_B}.$$

Therefore

$$\frac{a_{Al}}{a_{Bl}} \le \frac{a_{Al^*}}{a_{Bl^*}},$$

which contradicts the assumption $\frac{a_{Al}}{a_{Bl}} > \frac{a_{Al^*}}{a_{Bl^*}}$.

Therefore in equilibrium there is some good l^* such that

(i) if $l < l^*$, then only country A produces good l, and

(ii) if $l > l^*$, then only country B produces good l.

So in principle, you can compute the equilibrium by the following algorithm.

(i) Given prices $p_1, p_2, \dots, p_L$ and wages w_A, w_B, for each country i solve the utility maximization problem

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \le w_i e_i. \end{aligned}$$

Let the demand be $x_i(p, w_i e_i)$.

(ii) For each good $l^* = 1, \dots, L$, do the following.

(a) Given the wages w_A, w_B, compute the goods' prices by

$$p_l a_{il} = w_i \iff p_l = \begin{cases} \frac{w_A}{a_{Al}}, & (l < l^*) \\ \frac{w_A}{a_{Al}} = \frac{w_B}{a_{Bl}}, & (l = l^*) \\ \frac{w_B}{a_{Bl}}. & (l > l^*) \end{cases}$$

Since we can always normalize one price, set $p_{l^*} = 1$. Looking at the case $l = l^*$, then we have $w_A = a_{Al^*}$ and $w_B = a_{Bl^*}$. Then all other prices are pinned down by the cases $l < l^*$ and $l > l^*$. Namely, we have

$$p_l = \begin{cases} \frac{a_{Al^*}}{a_{Al}}, & (l < l^*) \\ 1, & (l = l^*) \\ \frac{a_{Bl^*}}{a_{Bl}}. & (l > l^*) \end{cases}$$

(b) Plug in these prices and wages into the formula for the demand $x_i(p, w_i e_i)$ and compute the demand of country A and B. Let x_{il} be the demand of good l by country i.

(c) Country A must produce the entire amount of goods $l < l^*$. So the labor input for good l must be

$$x_{Al} + x_{Bl} = a_{Al} e_{Al} \iff e_{Al} = \frac{x_{Al} + x_{Bl}}{a_{Al}}.$$

Compute total labor inputs except good l^*, that is,

$$\sum_{l=1}^{l^*-1} e_{Al}.$$

If this number exceeds total labor endowment e_A, it cannot be an equilibrium. So stop and go to the next l^*.

Similarly, country B must produce the entire amount of goods $l > l^*$, so the labor input for good l must be

$$x_{Al} + x_{Bl} = a_{Bl} e_{Bl} \iff e_{Bl} = \frac{x_{Al} + x_{Bl}}{a_{Bl}}.$$

Compute total labor inputs except good l^*, that is,

$$\sum_{l=l^*+1}^{L} e_{Bl}.$$

If this number exceeds total labor endowment e_B, it cannot be an equilibrium. So stop and go to the next l^*.

(iii) Repeat the above step until you find a good l^* such that

$$\sum_{l=1}^{l^*-1} e_{Al} \le e_A \text{ and } \sum_{l=l^*+1}^{L} e_{Bl} \le e_B.$$

This good l^* gives the equilibrium. The labor input for producing good l^* is

$$e_A - \sum_{l=1}^{l^*-1} e_{Al}$$

for country A and

$$e_B - \sum_{l=l^*+1}^{L} e_{Bl}$$

for country B.

In summary, Ricardo's model of international trade can be defined as follows. There are two countries and multiple consumption goods. Each country has a linear production technology to produce the goods from labor alone. In equilibrium, each country specializes in the production of goods that they have comparative advantage (goods that they are relatively more efficient to produce), except possibly one good that both countries produce. In equilibrium typically both countries will gain from trade (unless one country is so big that it is producing all goods, that is, either $l^* = 1$ or $l^* = L$). Contrary to intuition, free trade is especially beneficial for the small, inefficient country. This is because every country has a comparative advantage in at least one good, so the inefficient country can focus on the good that it has comparative advantage and import other goods much cheaper than if it had relied on domestic production.

9.3 Free trade in small open economies

From the above discussion, you might get an impression that free trade is great. But this is not necessarily the case. Consider a country with two agents, $i = 1, 2$. Suppose that there are two goods and the endowment is $e_1 = (9, 1)$ and $e_2 = (1, 9)$. Everybody has the utility function

$$u(x_1, x_2) = x_1 x_2,$$

which is the same as the Cobb-Douglas utility function

$$u(x_1, x_2) = \frac{1}{2} \log x_1 + \frac{1}{2} \log x_2.$$

Since the two agents are symmetric in endowment and the utility function is symmetric in the two goods, it should not be hard to guess that the autarky equilibrium is $(p_1, p_2) = (1, 1)$ and consumption is $x_1 = x_2 = (5, 5)$. The utility level is $5 \times 5 = 25$.

Now suppose that this country opens up to international trade and the world price becomes $(p_1, p_2) = (1, 2)$. What would happen to the consumption of each agent? By the Cobb-Douglas formula, letting w_i be the wealth of agent i, the consumption is

$$(x_{i1}, x_{i2}) = \left(\frac{w_i}{2p_1}, \frac{w_i}{2p_2} \right).$$

Under the new price, the wealth of agent 1 is $1 \times 9 + 2 \times 1 = 11$. Using this formula, the consumption of agent 1 is

$$(x_{11}, x_{12}) = \left(\frac{11}{2}, \frac{11}{4} \right).$$

Similarly, the wealth of agent 2 is $1 \times 1 + 2 \times 9 = 19$ and the consumption is

$$(x_{21}, x_{22}) = \left(\frac{19}{2}, \frac{19}{4}\right).$$

The utility of agent 1 becomes

$$u_1 = \frac{11}{2} \times \frac{11}{4} = \frac{121}{8} = 15.125 < 25$$

and the utility of agent 2 becomes

$$u_2 = \frac{19}{2} \times \frac{19}{4} = \frac{361}{8} = 45.125 > 25.$$

Therefore agent 2 gains from trade but agent 1 loses.

What has happened? The point is that after free trade, good 2 has become relatively more expensive than good 1. Since agent 2 has a large endowment of good 2, he has become rich. On the other hand, since agent 1 has a large endowment of the cheap good 1 but only a small endowment of the expensive good 2, he has become poor. Thus free trade is not necessarily Pareto improving when residents of a country have different endowments.

Should the government of the country be concerned about the potential loss of agent 1 and restrict trade? No. There is a way to make everybody better off. At the world price $(p_1, p_2) = (1, 2)$, the old consumption of each agent $(5, 5)$ has value $1 \times 5 + 2 \times 5 = 15$. On the other hand, the value of the endowment of each agent is 11 and 19, as computed above. So if the government imposes a direct tax on agent 2 and transfer the revenue to agent 1, we can make both agents better off. Suppose, for instance, that the government impose tax $t_2 = 4$ on agent 2 and $t_1 = -4$ on agent 1. Then both agents will have wealth 15 after the transfer. Given world price $(p_1, p_2) = (1, 2)$ and the Cobb-Douglas formula, both agents can now consume

$$(x_{i1}, x_{i2}) = \left(\frac{15}{2}, \frac{15}{4}\right),$$

which gives utility

$$\frac{15}{2} \times \frac{15}{4} = \frac{225}{8} = 28.125 > 25.$$

Thus both agents are better off than autarky, after free trade and the direct tax system.

The above example can be generalized as follows. Consider a country with I agents with endowments (e_i) and utility functions (u_i). Let $\{p^a, (x_i^a)\}$ be the autarky equilibrium. The following theorem shows that for a "small" country, unilateral free trade is Pareto improving after appropriate transfers.

Theorem 9.1. *Suppose that the country is small and therefore it takes world price p as given. Then there exist transfer payments (t_i) such that the free trade allocation weakly Pareto dominates the autarky allocation. More precisely, if x_i^f solves*

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \leq p \cdot e_i - t_i, \end{aligned}$$

then the free trade allocation (x_i^f) weakly Pareto dominates the autarky allocation (x_i^a).

Proof. By market clearing, we have $\sum_{i=1}^{I} x_i^a \le \sum_{i=1}^{I} e_i$. Therefore, for each i we can take a bundle y_i such that $y_i \ge x_i^a$ and $\sum_{i=1}^{I} y_i = \sum_{i=1}^{I} e_i$. Choose t_i so as to make the bundle y_i just affordable at the world price, so

$$p \cdot y_i = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - y_i).$$

By construction, we have

$$\sum_i t_i = \sum_i p \cdot (e_i - y_i) = p \cdot \sum_i (e_i - y_i) = 0,$$

so the transfer payments (t_i) are budget balanced. By the definition of y_i, we have

$$p \cdot x_i^a \le p \cdot y_i = p \cdot e_i - t_i,$$

so the bundle x_i^a is affordable at world price p. Therefore $u_i(x_i^f) \ge u_i(x_i^a)$ for all i, and (x_i^f) weakly Pareto dominates (x_i^a). □

9.4 Free trade in general equilibrium

The above theorem shows that free trade is always good for small countries, at least after appropriate direct taxes/subsidies. This is an example of *partial equilibrium analysis*: we are fixing the world price and ignoring the general equilibrium effect of a country opening up to trade. If a country is big, it will affect the world price by opening up to free trade. Since the transfers (t_i) and the world price p will interact with each other, the above proof cannot be applied.

However, even if countries are big, we can show that a *multilateral free trade* is efficient and Pareto improving. Consider a world consisting of I agents with endowments (e_i). Suppose that there are C countries indexed by $c = 1, \dots, C$. Let I_c be the set of residents in country c.

Theorem 9.2 (Efficiency of free trade with transfers)**.** *There exist transfer payments that are budget-feasible within each country such that the free trade equilibrium is efficient and weakly Pareto dominates the autarky equilibrium.*

More precisely, let $\mathcal{E}_c = \{I_c, (e_i), (u_i)\}$ be the economy of country c and $(p^c, (x_i^a)_{i \in I_c})$ be an autarky equilibrium in country c. Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be the world economy, where u_i is continuous, quasi-concave, and locally non-satiated. Then there exist a price vector p, an allocation (x_i^f), and transfer payments (t_i) such that

(i) *$(p, (x_i^f), (t_i))$ is a free trade equilibrium with transfer payments,*

(ii) *for each country c, transfer payments are budget-feasible, so $\sum_{i \in I_c} t_i = 0$,*

(iii) *the free trade allocation (x_i^f) weakly Pareto dominates the autarky allocation (x_i^a), that is, $u_i(x_i^f) \ge u_i(x_i^a)$ for all i, and*

(iv) *the free trade allocation (x_i^f) is Pareto efficient.*

Proof. By market clearing in country c, we have $\sum_{i \in I_c}(x_i^a - e_i) \leq 0$. For each i, take y_i such that $y_i \geq x_i^a$ and $\sum_{i \in I_c}(y_i - e_i) = 0$. Let $\mathcal{E}' = \{I, (y_i), (u_i)\}$ be a hypothetical world economy where agent i starts with the initial endowment y_i that is weakly larger than the autarky equilibrium allocation. Let $\left\{p, (x_i^f)\right\}$ be the equilibrium for the world economy $\mathcal{E}'$. By the first welfare theorem, (x_i^f) is efficient. To support this allocation in free trade with transfer payments in the original economy, let $t_i = p \cdot (e_i - y_i)$. By the definition of y_i, as in the proof of the previous theorem, we have $\sum_{i \in I_c}^{I} t_i = 0$. Therefore the government budget balances within each country. Since $\left\{p, (x_i^f)\right\}$ is a competitive equilibrium of $\mathcal{E}'$ (which has endowments y_i) and $p \cdot y_i = p \cdot e_i - t_i$ by the definition of t_i, it follows that $(p, (x_i^f), t_i)$ is an equilibrium with transfer payments for the economy $\mathcal{E}$. Since $x_i^a \leq y_i$ for all i and y_i is affordable after the transfer, so is the autarky consumption x_i^a. Therefore $u_i(x_i^f) \geq u_i(x_i^a)$ for all i, and the free trade (with transfer) allocation (x_i^f) weakly Pareto dominates the autarky allocation (x_i^a). □

9.5 Trade costs

So far we have assumed that goods can be freely transported across countries. But in reality there are costs to transport goods from one location to another. How should we deal with trade costs? Surprisingly, all we need to do is to redefine the goods.

Remember that in general equilibrium theory, goods are distinguished not only by physical properties but also by time, location, and states in which the good can be consumed. So a banana in Canada is different from a banana in Ecuador. How should we modify the model, then? Conceptually, that is not difficult. Suppose that it takes one unit of petrol to ship one unit of banana from Ecuador to Canada. We can interpret transportation as a technology that turns some goods into others; here, one unit of banana in Ecuador and one unit of petrol in Ecuador is transformed into one unit of banana in Canada. Then a model of international trade with transportation costs is merely a general equilibrium with many goods and many production technologies!

For example, suppose that there are I agents, C countries, and L physical goods (no production, for simplicity). Reinterpret the economy such that there are LC goods, where good (l, c) is physical good l in country c. Each agent gets utility only from consumption of good available in the country he or she lives in. Country c has access to a transportation technology set $Y_c \subset \mathbb{R}^{LC}$. Thus the general equilibrium theory requires no modification—a general equilibrium model of international trade with transportation cost is a special case of a bigger general equilibrium model with production.

Notes

Theorem 9.1 is due to Samuelson (1939). Theorem 9.2 is due to Grandmont and McFadden (1972).

Chapter 10

Finance

Recall that a general equilibrium model with uncertainty becomes a model of finance when we distinguish goods by the states in which they are available. I first introduce no-arbitrage asset pricing, whose goal is to compute the prices of some assets given the prices of other assets. This model is useful for computing the prices of derivatives. We can derive the capital asset pricing model by studying a general equilibrium model with HARA and quadratic utility.

10.1 No-arbitrage asset pricing

Consider an economy with two periods, denoted by $t = 0, 1$. Suppose that at $t = 1$ the state of the economy can be one of $s = 1, \dots, S$. There are J assets in the economy, indexed by $j = 1, \dots, J$. One share of asset j trades for price q_j at time 0 and pays A_{sj} in state s. (It can be $A_{sj} < 0$, in which case the holder of one share of asset j must deliver $-A_{sj} > 0$ in state s.) Let $q = (q_1, \dots, q_J)$ the vector of asset prices and $A = (A_{sj})$ be the matrix of asset payoffs. Define

$$W = W(q, A) = \begin{bmatrix} -q' \\ A \end{bmatrix}$$

be the $(1 + S) \times J$ matrix of net payments of one share of each asset in each state. Here, state 0 is defined by time 0 and the presence of $-q = (-q_1, \dots, -q_J)$ means that in order to receive A_{sj} in state s one must purchase one share of asset j at time 0, thus paying q_j (receiving $-q_j$).

Let $\theta \in \mathbb{R}^J$ be a *portfolio*. (θ_j is the number of shares of asset j an investor buys. $\theta_j < 0$ corresponds to shortselling.) The net payments of the portfolio θ is the vector

$$W\theta = \begin{bmatrix} -q'\theta \\ A\theta \end{bmatrix} \in \mathbb{R}^{1+S}.$$

Here the investor pays $q'\theta$ at $t = 0$ for buying the portfolio θ, and receives $(A\theta)_s$ in state s at $t = 1$.

Let $\langle W \rangle = \left\{ W\theta \,\middle|\, \theta \in \mathbb{R}^J \right\} \subset \mathbb{R}^{1+S}$ be the set of payoffs generated by all portfolios, called the *asset span*. We say that the asset span $\langle W \rangle$ exhibits *no-arbitrage* if

$$\langle W \rangle \cap \mathbb{R}^{1+S}_+ = \{0\}.$$

That is, it is impossible to find a portfolio that pays a non-negative amount in every state and a positive amount in at least one state. Then we can show the following theorem, due to Harrison and Kreps (1979).

Theorem 10.1 (Fundamental Theorem of Asset Pricing)**.** *The asset span $\langle W\rangle$ exhibits no-arbitrage if and only if there exists $p \in \mathbb{R}^S_{++}$ such that $[1, p']W = 0$.*

In this case, the asset prices are given by

$$q_j = \sum_{s=1}^{S} p_s A_{sj}.$$

$p_s > 0$ is called the state price *in state s.*

Proof. Suppose that such a p exists. If $0 \neq w = (w_0, \dots, w_S) \in \mathbb{R}^{1+S}_{+}$, then

$$[1, p']w = w_0 + \sum_{s=1}^{S} p_s w_s > 0,$$

so $w \notin \langle W\rangle$ because $[1, p']W = 0$. This shows $\langle W\rangle \cap \mathbb{R}^{1+S}_{+} = \{0\}$.

Conversely, suppose that there is no arbitrage. Then $\langle W\rangle \cap \Delta = \emptyset$, where $\Delta = \left\{ w \in \mathbb{R}^{1+S}_{+} \,\middle|\, \sum_{s=0}^{S} w_s = 1 \right\}$ is the unit simplex. Clearly $\langle W\rangle, \Delta$ are convex and nonempty, and Δ is compact. By the (strong version of) separating hyperplane theorem, we can find $0 \neq \lambda \in \mathbb{R}^{1+S}$ such that

$$\langle \lambda, w\rangle < \langle \lambda, d\rangle \tag{10.1}$$

for any $w \in \langle W\rangle$ and $d \in \Delta$. Let us show that $\lambda' W = 0$. Suppose not. Consider the portfolio $\theta = \alpha W'\lambda \in \mathbb{R}^J$, where $\alpha > 0$. Then by (10.1), for $w = W\theta$ we obtain

$$\langle \lambda, d\rangle > \langle \lambda, w\rangle = \langle \lambda, W(\alpha W'\lambda)\rangle = \alpha\lambda' W W'\lambda = \alpha \left\|\lambda' W\right\|^2 \to \infty$$

as $\alpha \to \infty$ because $\lambda' W \neq 0$, which is a contradiction. Therefore $\lambda' W = 0$, so $\langle \lambda, w\rangle = 0$ for all $w \in \langle W\rangle$. Then (10.1) becomes

$$0 < \langle \lambda, d\rangle$$

for all $d \in \Delta$. Letting $d = e_s$ (unit vector) for $s = 0, 1, \dots, S$, we get $\lambda_s > 0$. Dividing both sides of $\lambda' W = 0$ by $\lambda_0 > 0$ and letting $p_s = \lambda_s/\lambda_0$ for $s = 1, \dots, S$, the vector $p = (p_1, \dots, p_S)$ satisfies $p \gg 0$ and $[1, p']W = 0$. Writing down this equation component-wise, we get $q_j = \sum_{s=1}^{S} p_s A_{sj}$. $\square$

Since $p_s > 0$ for all s, we have $\sum_{s=1}^{S} p_s > 0$. Since the risk-free asset pays 1 in every state, its price is

$$\frac{1}{1+r} = \sum_{s=1}^{S} p_s > 0.$$

Letting $\nu_s = p_s / \sum_s p_s > 0$, we have $\sum_s \nu_s = 1$ and

$$q_j = \frac{1}{1+r} \sum_{s=1}^{S} \nu_s A_{sj} = \frac{1}{1+r} \tilde{\mathrm{E}}[A_{sj}].$$

Therefore the asset price is the discounted expected payoff of the asset using the *risk-neutral* probability measure $\{\nu_s\}$. This formula is useful for computing option prices in continuous time. For more details, see Duffie (2001).

Letting π_s be the objective probability of state s and $m_s = \frac{p_s}{\pi_s}$, we have

$$q_j = \sum_{s=1}^{S} p_s A_{sj} = \sum_{s=1}^{S} \pi_s m_s A_{sj} = \mathrm{E}[mA_j].$$

The random variable m is called the *stochastic discount factor*, or SDF for short. Letting $R_j = A_j/q_j$ be the gross return of the asset, we have

$$\mathrm{E}[mR_j] = 1$$

for any asset. The risk-free rate R_f satisfies $\mathrm{E}[mR_f] = 1 \iff R_f = 1/\mathrm{E}[m]$. Using the definition of the covariance,

$$\mathrm{Cov}[X, Y] = \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y],$$

we obtain

$$\begin{aligned} 0 = \mathrm{E}[m(R_j - R_f)] &= \mathrm{E}[m](\mathrm{E}[R_j] - R_f) + \mathrm{Cov}[m, R_j - R_f] \\ \iff \mathrm{E}[R_j] - R_f &= -\frac{1}{\mathrm{E}[m]}\mathrm{Cov}[m, R_j - R_f] \\ &= -R_f\,\mathrm{Cov}[m, R_j], \end{aligned}$$

which is known as the covariance pricing formula.

10.2 Option pricing

As an application of the no-arbitrage asset pricing, in this section I explain the binomial option pricing model of Cox et al. (1979).

Consider a T period economy, and time is indexed by $t = 0, 1, \dots, T$. Suppose that there are two assets, a stock and a bond. The gross risk-free rate is constant at R, and the stock price at time t is denoted by S_t, which is a random variable. Assume that the stock can go up or down, so

$$S_{t+1} = \begin{cases} US_t, & \text{(if stock goes up)} \\ DS_t, & \text{(if stock goes down)} \end{cases}$$

where $U > R > D$. Question: what is the price of a call option with strike K?

This question seems hopeless to answer since we have not even specified the probabilities of up and down. It turns out that the answer does not depend on the probability, so an optimist and a pessimist will still agree on the price of the option.

Recall that a call (put) option with strike price K and maturity T is a contract such that the holder has the right (but not the obligation) to buy (sell) the stock at price K until the maturity. The act of buying/selling the stock at the specified price is called *exercising*. If the investor can exercise the option at any time on or before maturity, it is called *American*. If the option can be exercised only at maturity, it is called *European*. For more details see textbooks such as Shreve (2004).

Let us compute the price of a European call option. First, consider the simplest case where there is no time, so $T = 0$. Let C be the call price. If the investor exercises the option, he gets $S_0 - K$ by buying the stock at strike price K and selling at the market value S_0. If the investor does not exercise the option, it expires, and he gets 0. A rational investor will choose the better alternative, so

$$C = \max\{S_0 - K, 0\}.$$

Next, consider the case with one period to go. If the stock price goes up at $t = 1$, by the above argument the option price becomes $C_u = \max\{US_0 - K, 0\}$. Similarly, in the down state at $t = 1$, the option price is $C_d = \max\{DS_0 - K, 0\}$. Letting p_s be the state price of state $s = u, d$, by no-arbitrage we have

$$C = p_u C_u + p_d C_d.$$

Therefore it remains to compute p_u, p_d. To this end we use the no-arbitrage condition for the stock and bond. Since the stock price is S_0 at $t = 0$, and it is US_0 in the up state and DS_0 in the down state, we have

$$S_0 = p_u US_0 + p_d DS_0 \iff 1 = p_u U + p_d D.$$

Since the risk-free asset pays R in all states for one unit of money invested, we have

$$1 = p_u R + p_d R.$$

Solving the system of two linear equations in two unknowns, we get

$$\begin{bmatrix} p_u \\ p_d \end{bmatrix} = \frac{1}{R} \begin{bmatrix} p \\ 1-p \end{bmatrix},$$

where $p = \frac{R-D}{U-D}$. Therefore the call price is

$$C = \frac{1}{R}(pC_u + (1-p)C_d) = \frac{1}{1+r}\tilde{\mathrm{E}}[C_s],$$

where r is the net risk-free rate and $\tilde{\mathrm{E}}$ denotes the expectation under the risk-neutral probability p.

The general case is completely analogous. If there are T periods to go, payoffs must be discounted by R^T. Since there are two states (up or down) following any state, the risk-neutral probability is $(p, 1-p)$ each, the risk-neutral probability at T is a binomial distribution with probability p. The probability that there are n up states is $\binom{T}{n} p^n (1-p)^{T-n}$, and in this case the final stock price is $S_T = U^n D^{T-n}$. Thus the price of a European call option must be

$$C = \frac{1}{R^T} \sum_{n=0}^{T} \binom{T}{n} p^n (1-p)^{T-n} \max\left\{U^n D^{T-n} S_0 - K, 0\right\}.$$

The pricing of European put option is also analogous. Recalling that the payoff of a put when the stock price is S and the strike is K is $P = \max\{K - S, 0\}$, by the same argument the price of a European put is

$$P = \frac{1}{R^T} \sum_{n=0}^{T} \binom{T}{n} p^n (1-p)^{T-n} \max\left\{K - U^n D^{T-n} S_0, 0\right\}.$$

An important property of the European options is the put-call parity:

$$C - P = S_0 - KR^{-T}$$

always. Therefore if we know the call price, we can compute the put price by $P = C - S_0 + KR^{-T}$, so we do not need to repeat the calculation. To prove the put-call parity, note that the payoff of a call is $\max\{S_T - K, 0\}$, and that of the put is $\max\{K - S_T, 0\}$. But since

$$\begin{aligned}\max\{S_T - K, 0\} - \max\{K - S_T, 0\} &= \max\{S_T - K, 0\} + \min\{S_T - K, 0\} \\ &= S_T - K,\end{aligned}$$

if someone buys one call and short one put, the terminal payoff is equal to that of holding the stock and paying K at the terminal date. The present value of this portfolio is exactly $S_0 - KR^{-T}$, so the put-call parity holds.

10.3 Capital Asset Pricing Model (CAPM)

No-arbitrage asset pricing tells us that under fairly weak conditions (absence of arbitrage), there exist state prices. This theorem enables us to compute the prices of derivatives given the prices of fundamental assets. However, no-arbitrage asset pricing does not tell us the prices of fundamental assets because it does not pin down the state prices in general. By embedding no-arbitrage asset pricing into a general equilibrium model, we can say something about asset prices. This is the capital asset pricing model (CAPM).

Theorem 10.2. *Let $\mathcal{E} = \{I, (e_i), (U_i)\}$ be an Arrow-Debreu economy with two periods, denoted by $t = 0, 1$, and S states at $t = 1$ (state s occurs with probability $\pi_s > 0$). Suppose that agent i has utility function*

$$U_i(x_0, \dots, x_S) = u_i(x_0) + \beta\,\mathrm{E}[u_i(x_s)] = u_i(x_0) + \beta \sum_{s=1}^{S} \pi_s u_i(x_s),$$

where $\beta > 0$ is the (common) discount factor and $u_i(x)$ is a HARA Bernoulli utility function with parameters (a, b_i) (so a is common across agents). Normalizing the price of $t = 0$ good as $p_0 = 1$, the state price is then

$$p_s = \beta\pi_s \left(\frac{ae_s + b}{ae_0 + b}\right)^{-1/a}.$$

Proof. By the same argument as in earlier aggregation results, we can prove that the economy is equivalent to one with a HARA representative agent with parameters (a, b), where $b = \sum_{i=1}^{I} b_i$, who consumes the aggregate endowment.

Recall that the HARA utility has the functional form

$$u(x) = \frac{1}{a-1}(ax + b)^{1-1/a}.$$

Therefore the marginal utility is $u'(x) = (ax + b)^{-1/a}$. In equilibrium, the marginal rate of substitution between $t = 0$ and state s must be equal to the price ratio, so normalizing the price of $t = 0$ good to be $p_0 = 1$, we obtain the state price

$$p_s = \frac{p_s}{p_0} = \beta\pi_s \left(\frac{ae_s + b}{ae_0 + b}\right)^{-1/a}. \qquad \square$$

An immediate consequence of Theorem 10.2 is the following Mutual Fund Theorem (Cass and Stiglitz, 1970; Merton, 1971).

Corollary 10.3 (Mutual Fund Theorem)**.** *Let everything be as in Theorem 10.2. Then any agent's consumption at $t = 1$ can be replicated just by the aggregate endowment ("stock market") and the vector of ones $1 = (1, \dots, 1)'$ ("risk-free asset").*

Proof. Letting $\lambda_i > 0$ be the Lagrange multiplier of agent i, by the first-order condition we obtain

$$\begin{aligned} \beta\pi_s(ax_s + b)^{-1/a} &= \lambda_i p_s = \lambda_i \beta \pi_s \left(\frac{ae_s + b}{ae_0 + b}\right)^{-1/a} \\ \iff ax_s + b &= \lambda_i^{-a} \frac{ae_s + b}{ae_0 + b} \\ \iff x_s &= \lambda_i^{-a} \frac{1}{ae_0 + b} e_s + \frac{b}{a}\left(\lambda_i^{-a}\frac{1}{ae_0 + b} - 1\right). \end{aligned}$$

Therefore there exist constants $\theta_i > 0$, $\phi_i \in \mathbb{R}$ such that $x_i = \theta_i e + \phi_i 1$, where $x_i = (x_{i1}, \dots, x_{iS})'$ and $e = (e_1, \dots, e_S)'$, so agent i's consumption at $t = 1$ can be spanned by the stock and the risk-free asset. □

The Mutual Fund Theorem has an enormous practical implication: no matter what your risk attitude is or traded assets are, the optimal portfolio is a combination of the aggregate stock market and the risk-free asset. Thus all you need to decide is how much to invest in each asset. Influenced by this theorem, John Bogle founded the Vanguard Group in 1974 and started offering the first index fund in 1975.[1]

By specializing Theorem 10.2 to quadratic utility, we obtain the capital asset pricing model (CAPM). Let us model the aggregate stock market by an asset that pays off the aggregate endowment, so the aggregate stock price q_m is the price of the payoff $(0, e_1, \dots, e_S)$ and its gross return is the vector $R_m = (e_1/q_m, \dots, e_S/q_m)$.

Theorem 10.4. *Let everything be as in Theorem 10.2 and $a = -1$, so agents have quadratic utility. Let R_f be the gross risk-free rate and R_j be the gross return of any asset j. Then the following covariance pricing formula holds:*

$$\mathrm{E}[R_j] - R_f = \beta_j(\mathrm{E}[R_m] - R_f),$$

where $\beta_j = \mathrm{Cov}[R_m, R_j]/\mathrm{Var}[R_m]$ is the market beta *of asset j.*

Proof. By previous results, the stochastic discount factor in state s is

$$m_s = \frac{p_s}{\pi_s} = \beta\left(\frac{ae_s + b}{ae_0 + b}\right)^{-1/a} = \beta\frac{b - e_s}{b - e_0},$$

where we have used $a = -1$. Since β, b, e_0 are all constants and the gross market return $R_{ms} = e_s/q_m$ is proportional to e_s, we can write this as

$$m_s = A - BR_{ms},$$

[1] https://en.wikipedia.org/wiki/Index_fund

where $B > 0$. Therefore the stochastic discount factor is linear in the market return. By the covariance pricing formula, we obtain

$$\mathrm{E}[R_j] - R_f = -R_f \operatorname{Cov}[m, R_j] = R_f B \operatorname{Cov}[R_m, R_j].$$

Since this equation is true for any asset, in particular we can set $R_j = R_m$. Therefore

$$\mathrm{E}[R_m] - R_f = R_f B \operatorname{Cov}[R_m, R_m] = R_f B \operatorname{Var}[R_m].$$

Eliminating $B > 0$ from these two equations, we obtain

$$\mathrm{E}[R_j] - R_f = \beta_j (\mathrm{E}[R_m] - R_f),$$

where $\beta_j = \operatorname{Cov}[R_m, R_j] / \operatorname{Var}[R_m]$. $\square$

The quantity

$$\beta_j = \frac{\operatorname{Cov}[R_m, R_j]}{\operatorname{Var}[R_m]}$$

is called the *beta* of the asset. By definition, the beta of the market return is $\beta_m = 1$. Beta measures the *market risk* of an asset.

Rewriting the covariance pricing formula, we obtain

$$\mathrm{E}[R_j] = R_f + \beta_j (\mathrm{E}[R_m] - R_f).$$

The theoretical linear relationship between β_j and $\mathrm{E}[R_j]$ is called the *security market line* (SML) (Figure 10.1). An asset above (below) the security market line, that is,

$$\mathrm{E}[R_j] > (<) R_f + \beta_j (\mathrm{E}[R_m] - R_f)$$

is undervalued (overvalued) because the expected return is higher (lower) than predicted.

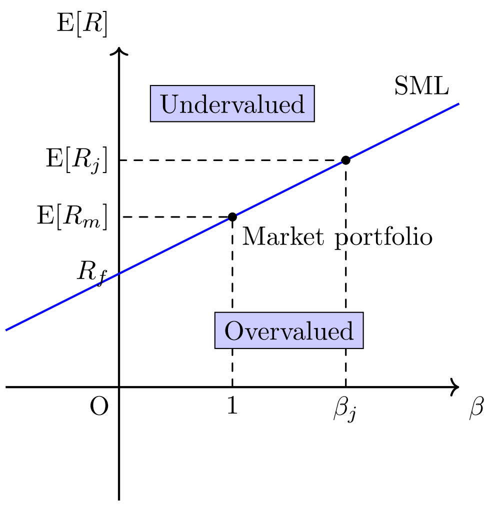

Figure 10.1: Security market line.

Bibliography

Arrow, K. J. (1951). "An Extension of the Basic Theorems of Classical Welfare Economics". In: *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*. Ed. by J. Neyman. University of California Press.

Arrow, K. J. (1964). "The Role of Securities in the Optimal Allocation of Riskbearing". *Review of Economic Studies* 31.2, 91–96. DOI: 10.2307/2296188.

Arrow, K. J. and G. Debreu (1954). "Existence of an Equilibrium for a Competitive Economy". *Econometrica* 22.3, 265–290. DOI: 10.2307/1907353.

Berge, C. (1963). *Topological Spaces*. Edinburgh: Oliver & Boyd.

Cass, D. and J. E. Stiglitz (1970). "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds". *Journal of Economic Theory* 2.2, 122–160. DOI: 10.1016/0022-0531(70)90002-5.

Chipman, J. S. (1974). "Homothetic Preferences and Aggregation". *Journal of Economic Theory* 8.1, 26–38. DOI: 10.1016/0022-0531(74)90003-9.

Chipman, J. S. and J. C. Moore (1979). "On Social Welfare Functions and the Aggregation of Preferences". *Journal of Economic Theory* 21.1, 111–139. DOI: 10.1016/0022-0531(79)90009-7.

Cox, J. C., S. A. Ross, and M. Rubinstein (1979). "Option Pricing: A Simplified Approach". *Journal of Financial Economics* 7.3, 229–263. DOI: 10.1016/0304-405X(79)90015-1.

Debreu, G. (1951). "The Coefficient of Resource Utilization". *Econometrica* 19.3, 273–292. DOI: 10.2307/1906814.

Debreu, G. (1959). *Theory of Value*. Cowles Foundation Monograph 17. New Haven: Yale University Press.

Debreu, G. (1970). "Economies with a Finite Set of Equilibria". *Econometrica* 38.3, 387–392. DOI: 10.2307/1909545.

Debreu, G. (1974). "Excess Demand Functions". *Journal of Mathematical Economics* 1.1, 15–21. DOI: 10.1016/0304-4068(74)90032-9.

Duffie, D. (2001). *Dynamic Asset Pricing Theory*. 3rd ed. Princeton University Press.

Duffie, D. and H. Sonnenschein (1989). "Arrow and General Equilibrium Theory". *Journal of Economic Literature* 27.2, 565–598.

Geanakoplos, J. (1987). "The Arrow-Debreu Model of General Equilibrium". In: *The New Palgrave Dictionary of Money and Finance*. Ed. by J. Eatwell, M. Milgate, and P. Newman. Vol. 1. London: Macmillan, 116–124.

Geanakoplos, J. and K. J. Walsh (2018a). "Inefficient Liquidity Provision". *Economic Theory* 66.1, 213–233. DOI: 10.1007/s00199-017-1059-7.

Geanakoplos, J. and K. J. Walsh (2018b). "Uniqueness and Stability of Equilibrium in Economies with Two Goods". *Journal of Economic Theory* 174, 261–272. DOI: 10.1016/j.jet.2017.12.005.

Gorman, W. M. (1953). "Community Preference Fields". *Econometrica* 21.1, 63–80. DOI: 10.2307/1906943.

Grandmont, J.-M. and D. McFadden (1972). "A Technical Note on Classical Gains from Trade". *Journal of International Economics* 2.2, 109–125. DOI: 10.1016/0022-1996(72)90029-3.

Harrison, J. M. and D. M. Kreps (1979). "Martingales and Arbitrage in Multiperiod Securities Market". *Journal of Economic Theory* 20.3, 381–408. DOI: 10.1016/0022-0531(79)90043-7.

Hens, T. and A. Loeffler (1995). "Gross Substitution in Financial Markets". *Economics Letters* 49.1, 39–43. DOI: 10.1016/0165-1765(94)00647-K.

Kehoe, T. J. (1998). "Uniqueness and Stability". In: *Elements of General Equilibrium Analysis*. Ed. by A. Kirman. Wiley-Blackwell. Chap. 3, 38–87.

Lange, O. (1942). "The Foundations of Welfare Economics". *Econometrica* 10.3/4, 215–228. DOI: 10.2307/1905465.

Mantel, R. R. (1974). "On the Characterization of Aggregate Excess Demand". *Journal of Economic Theory* 7.3, 348–353. DOI: 10.1016/0022-0531(74)90100-8.

Mantel, R. R. (1976). "Homothetic Preferences and Community Excess Demand Functions". *Journal of Economic Theory* 12.2, 197–201. DOI: 10.1016/0022-0531(76)90073-9.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995). *Microeconomic Theory*. New York: Oxford University Press.

McKenzie, L. W. (1954). "On Equilibrium in Graham's Model of World Trade and Other Competitive Systems". *Econometrica* 22.2, 147–161. DOI: 10.2307/1907539.

Merton, R. C. (1971). "Optimum Consumption and Portfolio Rules in a Continuous-Time Model". *Journal of Economic Theory* 3.4, 373–413. DOI: 10.1016/0022-0531(71)90038-X.

Pratt, J. W. (1964). "Risk Aversion in the Small and in the Large". *Econometrica* 32.1-2, 122–136. DOI: 10.2307/1913738.

Samuelson, P. A. (1939). "The Gains from International Trade". *Canadian Journal of Economics and Political Science* 5.2, 195–205.

Shreve, S. E. (2004). *Stochastic Calculus for Finance II—Continuous-Time Models*. Springer Finance. New York: Springer.

Sonnenschein, H. (1972). "Market Excess Demand Functions". *Econometrica* 40.3, 549–563. DOI: 10.2307/1913184.

Sonnenschein, H. (1973). "Do Walras' Identity and Continuity Characterize the Class of Community Excess Demand Functions?" *Journal of Economic Theory* 6.4, 345–354. DOI: 10.1016/0022-0531(73)90066-5.

Starr, R. M. (2011). *General Equilibrium Theory: An Introduction*. 2nd ed. Cambridge University Press.

Toda, A. A. and K. J. Walsh (2017). "Edgeworth Box Economies with Multiple Equilibria". *Economic Theory Bulletin* 5.1, 65–80. DOI: 10.1007/s40505-016-0102-3.

Uzawa, H. (1962). "Walras' Existence Theorem and Brouwer's Fixed-Point Theorem". *Economic Studies Quarterly* 13, 59–62.

Weintraub, R. E. (2011). “Retrospectives: Lionel W. McKenzie and the Proof of the Existence of a Competitive Equilibrium”. *Journal of Economic Perspectives* 25.2, 199–215. DOI: 10.1257/jep.25.2.199.