Econ 200A Microeconomic Theory

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- This document compiles past exam questions.

1 Basics

1. Consider the CES utility function (with a slightly different parameterization)

$$u(x_1, x_2; \sigma) = (\alpha_1^\sigma x_1^{1-\sigma} + \alpha_2^\sigma x_2^{1-\sigma})^{\frac{1}{1-\sigma}},$$

   where $\alpha_1, \alpha_2, \sigma > 0$ are parameters.

   (a) Assume $\alpha_1 + \alpha_2 = 1$. Compute $\lim_{\sigma \to 1} u(x_1, x_2; \sigma)$ and show that the Cobb-Douglas utility function is a special case of CES (up to a monotonic transformation).

   (b) Compute

$$\lim_{\sigma \to \infty} u(x_1, x_2; \sigma)$$

   and show that the Leontief utility function is a special case of CES.

2. Consider an agent with CES utility function

$$u(x_1, x_2) = \frac{1}{1-\sigma}(\alpha_1^\sigma x_1^{1-\sigma} + \alpha_2^\sigma x_2^{1-\sigma}),$$

   where $\alpha_1, \alpha_2, \sigma > 0$ are parameters. (As usual, $\sigma = 1$ corresponds to Cobb-Douglas.) Let $w > 0$ be the wealth of the agent and $p_1, p_2 > 0$ be the price of goods. Compute the demand.

3. Consider an agent with the CES utility function

$$u(x_1, x_2) = \frac{1}{1-\sigma}(\alpha_1^\sigma x_1^{1-\sigma} + \alpha_2^\sigma x_2^{1-\sigma}).$$

   Let (e_1, e_2) be the initial endowment. Let $x_l(p_1, p_2)$ be the demand for good l, given prices.

   (a) Show that if $\sigma \leq 1$, then $x_1(p_1, p_2)$ is decreasing in p_1, so demand curves are downward sloping.

   (b) Show that if $\sigma = \infty$ (Leontief) and $\frac{e_1}{\alpha_1} > \frac{e_2}{\alpha_2}$, then $x_1(p_1, p_2)$ is increasing in p_1.

4. Consider an agent with additively separable utility function

$$U(x_1, x_2) = u_1(x_1) + u_2(x_2),$$

   where $u_l' > 0$ and $u_l'' < 0$ for $l = 1, 2$. Let $p_1, p_2 > 0$ be the price of goods and (e_1, e_2) be the initial endowment. Let

$$\gamma_l(x) = -\frac{x u_l''(x)}{u_l'(x)}$$

   be the relative risk aversion coefficient with respect to good l. Below, let (x_1, x_2) be the demand given the prices.

   (a) Let $p = p_1/p_2$ be the relative price of good 1. Show that $u_1'(x_1) = p u_2'(e_2 - p(x_1 - e_1))$.

   (b) Show that if $\gamma_2 \leq 1$, then $\frac{\partial x_1}{\partial p_1} < 0$, so the demand of good 1 is downward sloping.

5. Consider an economy with I agents and two goods. Suppose that agent i's utility function is

$$U_i(x_1, x_2) = \frac{1}{1-\gamma_i} x_1^{1-\gamma_i} + \beta_i \log x_2,$$

where $\beta_i, \gamma_i > 0$ are parameters. For arbitrary initial endowment $(e_{i1}, e_{i2})_{i \in I}$, show that there exists a unique equilibrium.

6. Consider an economy consisting of two agents, $i = A, B$, and two goods, $l = 1, 2$. The initial endowments are $e_A = a = (a_1, a_2) \gg 0$, $e_B = b = (b_1, b_2) \gg 0$, and agents have Cobb-Douglas utility functions

$$U_A(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2,$$
$$U_B(x_1, x_2) = \beta \log x_1 + (1 - \beta) \log x_2,$$

where $\alpha, \beta \in (0, 1)$. Below, normalize the price of good 2 to be $p_2 = 1$ and set $p_1 = p$.

   (a) Let $w = pa_1 + a_2$ be the wealth of agent A. Write down the Lagrangian of the utility maximization problem.
   (b) Derive the first-order conditions.
   (c) Express the demand of agent A using only α, a_1, a_2, p.
   (d) Compute the equilibrium price.
   (e) Show that if either (i) the supply of good 1 decreases (a_1 or b_1 goes down), (ii) the supply of good 2 increases (a_2 or b_2 goes up), or (iii) agents like good 1 more (α or β goes up), then the price of good 1 increases.

7. Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy and $\{p, (x_i)\}$ be an equilibrium, where $p = (p_1, \dots, p_L)$ is the price vector.

   (a) Assume $p_l < 0$. If an agent buys $\epsilon > 0$ of extra good l, how much more (or less) will his or her wealth be?
   (b) Prove that if we assume free disposal and at least one agent has a locally non-satiated utility function, then prices cannot be negative.

8. Consider an agent with a CES (c̲onstant e̲lasticity of s̲ubstitution) utility function

$$u(x_1, x_2) = \frac{1}{1-\sigma} \left(\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma} \right),$$

where $\alpha_1, \alpha_2 > 0$ and $1 \neq \sigma > 0$. Assume that the agent has wealth $w > 0$ and that the price each good is p_1, p_2.

   (a) Write down the Lagrangian for the utility maximization problem using the Lagrange multiplier $\lambda \geq 0$.
   (b) Derive the first order conditions with respect to x_1 and x_2.
   (c) Express $\frac{x_2}{x_1}$ as a function of $\frac{p_2}{p_1}$.
   (d) In general,

$$\varepsilon = -\frac{\mathrm{d} \log(x_2/x_1)}{\mathrm{d} \log(p_2/p_1)}$$

   is called the *elasticity of substitution* between good 1 and 2. (If you don't like the expression $\mathrm{d} \log(p_2/p_1)$, just set $t = \log(p_2/p_1)$ and think of $\log(x_2/x_1)$ as a function of t alone. "d" here refers to the derivative, of course.)
   Compute the elasticity of substitution for an agent with CES utility.

9. Consider an Arrow-Debreu economy with two agents indexed by $i = 1, 2$. Suppose that the utility functions are

$$\begin{aligned} U_1(x_1, x_2) &= \alpha \log x_1 + (1 - \alpha) \log x_2, \\ U_2(x_1, x_2) &= \min\{x_1, x_2\}, \end{aligned}$$

where $0 < \alpha < 1$ is a preference parameter. Let agent i's initial endowment be (e_{i1}, e_{i2}). Let $p_1 = 1$ and $p_2 = p$ be the prices.

(a) Compute each agent's demand for good 1, given p.

(b) Derive a necessary and sufficient condition such that p is an equilibrium price for some $e_{12} > 0$.

(c) Show that when agent 1 likes good 1 more (α increases), the relative price of good 1, which is $1/p$, increases (so p decreases).

10. Consider an economy with two goods and I agents. Agent i has endowment $(e_{i1}, e_{i2}) \gg 0$ and utility function

$$u_i(x_1, x_2) = \alpha_i \log x_1 + (1 - \alpha_i) \log x_2,$$

where $0 < \alpha_i < 1$. Let the prices be $p_1 = 1$ and $p_2 = p$.

(a) Write down the Lagrangian for the utility maximization problem of agent i.

(b) Using the first-order condition, express agent i's demand using p and the Lagrange multiplier.

(c) Express agent i's demand using only p and other exogenous parameters.

(d) Does this economy have an equilibrium? If so, is it unique?

11. Consider an economy with I agents and L goods. Suppose agent i has the Cobb-Douglas utility function

$$u_i(x) = \sum_{l=1}^{L} \alpha_{il} \log x_l,$$

where $x = (x_1, \dots, x_L)'$, $\alpha_{il} > 0$, and $\sum_{l=1}^{L} \alpha_{il} = 1$. Let $e_i \gg 0$ be the endowment of agent i and $e_l = \sum_{i=1}^{I} e_{il}$ be the aggregate endowment of good l.

(a) Let $p = (p_1, \dots, p_L)' \gg 0$ be the price vector. Compute the demand of agent i.

(b) Show that in equilibrium, it must be $p \gg 0$ and

$$p \cdot \sum_{i=1}^{I} \alpha_{il} e_i = p_l e_l, \quad l = 1, \dots, L.$$

(c) Define the vector $q = (q_1, \dots, q_L)' \in \mathbb{R}^L_{++}$ by $q_l = p_l e_l$ and the $L \times L$ matrix $B = (b_{lm})$ by

$$b_{lm} = \sum_{i=1}^{I} \alpha_{il} \frac{e_{im}}{e_m}.$$

Show that the equilibrium condition in the previous question is equivalent to $Bq = q$ and that the matrix B' is a positive transition probability matrix.

(d) Prove that a unique equilibrium exists.

12. Consider an economy with two physical goods (apples and bananas) and two periods denoted by $t = 1, 2$. There is only one agent (type) with utility

$$U(x_1, y_1, x_2, y_2) = u(x_1, y_1) + \beta u(x_2, y_2),$$

where

$$u(x, y) = \alpha \log x + (1 - \alpha) \log y$$

is the period utility function, $\beta > 0$ is the discount factor, and x_t, y_t denote the consumption of apples and bananas in period t. Let the endowment of apples and bananas in period t be a_t and b_t.

(a) Compute the gross real interest rate in units of apples.

(b) Compute the gross real interest rate in units of bananas.

(c) Under what condition do the two interest rates coincide? Is it surprising that the two rates are distinct except a very special case?

13. Prove that strong monotonicity implies weak monotonicity, which implies local non-satiation.

14. Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy with locally non-satiated utilities. Suppose that for some i, u_i is weakly monotonic. Show that if $\mathcal{E}$ has a competitive equilibrium, then it also has an equilibrium in which all markets clear exactly, that is,

$$\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i.$$

15. Consider an Arrow-Debreu economy with two agents denoted by $i = A, B$ and S states denoted by $s = 1, \dots, S$. Let $\pi_s > 0$ be the objective probability of state s, where $\sum_{s=1}^{S} \pi_s = 1$. Let $e_{is} > 0$ be agent i's initial endowment of good s. Suppose that the utility functions are given by

$$U_A(x) = \sum_{s=1}^{S} \pi_s u(x_s),$$
$$U_B(x) = \sum_{s=1}^{S} \pi_s x_s,$$

where $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. What is the most you can say about equilibrium prices and allocations?

16. The hyperbolic absolute risk aversion (HARA) utility function satisfies

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax + b},$$

where a, b are constants and we only consider the range of x such that $ax + b > 0$.

(a) Derive the functional form of u (up to a monotonic transformation) when $a = 0$.

(b) Derive the functional form of u (up to a monotonic transformation) when $a = 1$.

(c) Derive the functional form of u (up to a monotonic transformation) when $a \neq 0, 1$.

2 Quasi-linear model

1. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility function of each agent is

$$u_A(x_1, x_2) = x_1 + 2\sqrt{x_2},$$
$$u_B(x_1, x_2) = x_1 + 2\log x_2.$$

Suppose that the initial endowments are $(e_1^A, e_2^A) = (4, 5)$ and $(e_1^B, e_2^B) = (10, 3)$.

(a) Consider the problem of maximizing the sum of utilities,

$$\begin{aligned} \text{maximize} \quad & u_A(x_1^A, x_2^A) + u_B(x_1^B, x_2^B) \\ \text{subject to} \quad & x_1^A + x_1^B \le e_1^A + e_1^B, \\ & x_2^A + x_2^B \le e_2^A + e_2^B. \end{aligned}$$

Let $(x_1^A, x_2^A, x_1^B, x_2^B)$ be a solution. Compute x_2^A and x_2^B.

(b) Let the price be $p_1 = 1$ and $p_2 = p$. Compute the equilibrium price p and the allocation.

2. Consider an economy with two agents ($i = 1, 2$) and two goods ($l = 1, 2$). The utility functions are

$$u_1(x_1, x_2) = x_1 - \frac{9}{x_2},$$
$$u_2(x_1, x_2) = x_1 + 2\log x_2.$$

The initial endowments are $e_1 = (5, 1)$ and $e_2 = (4, 4)$. Compute the competitive equilibrium.

3. Consider an economy with two goods, 1 and 2, and two agents, A and B. The utility function of each agent is

$$u_A(x_1, x_2) = x_1 - \frac{8}{x_2},$$
$$u_B(x_1, x_2) = x_1 + 10\log x_2.$$

Suppose that the initial endowments are $(e_1^A, e_2^A) = (4, 4)$ and $(e_1^B, e_2^B) = (10, 3)$.

(a) Consider the problem of maximizing the sum of utilities,

$$\begin{aligned} \text{maximize} \quad & u_A(x_1^A, x_2^A) + u_B(x_1^B, x_2^B) \\ \text{subject to} \quad & x_1^A + x_1^B \le e_1^A + e_1^B, \\ & x_2^A + x_2^B \le e_2^A + e_2^B. \end{aligned}$$

Let $(x_1^A, x_2^A, x_1^B, x_2^B)$ be a solution. Compute x_2^A and x_2^B.

(b) Let the price be $p_1 = 1$ and $p_2 = p$. Compute the equilibrium price p and the allocation.

4. Consider a quasi-linear economy with two goods and two agents indexed by $i \in \{1, 2\}$. Agent i is endowed with (e_{i0}, e_{i1}) of each good and has utility function

$$u_i(x_0, x_1) = x_0 + \beta_i \log x_1,$$

where $\beta_i > 0$. Compute the Walrasian equilibrium.

5. Consider a quasi-linear economy with two goods and I agents indexed by i. Agent i is endowed with (e_{i0}, e_{i1}) of each good and has utility function

$$u_i(x_0, x_1) = x_0 + \beta_i \frac{x_1^{1-\sigma}}{1-\sigma},$$

where $\beta_i > 0$ and $\sigma > 0$. Compute the Walrasian equilibrium.

6. Let $\mathcal{E} = \{I, (e_{i0}, e_i), (u_i)\}$ be an economy with quasi-linear utility functions $u_i(x_0, x) = x_0 + \phi_i(x)$. Suppose that ϕ_i's are differentiable, strictly increasing, and strictly concave. Let $\mathcal{E}' = \{I, (e'_{i0}, e'_i), (u_i)\}$ be another economy with the same utility functions but potentially different endowments. Show that if $\sum_{i=1}^I e_i = \sum_{i=1}^I e'_i$, then the equilibria of $\mathcal{E}, \mathcal{E}'$ coincide in the sense that the prices and the allocation of the non-numéraire goods are the same.

3 Welfare theorems

1. Consider an economy with L goods and I agents. Agent i has endowment e_i and a locally nonsatiated utility function $u_i(x)$. Let p be the price vector.
   (a) What is the definition of local nonsatiation?
   (b) Suppose x_i solves the utility maximization problem

   $$\text{maximize } u_i(x) \text{ subject to } p \cdot x \leq p \cdot e_i.$$

   Explain why it must be the case that $p \cdot x_i = p \cdot e_i$.
   (c) What does it mean that an allocation (y_i) Pareto dominates the allocation (x_i)?
   (d) What does it mean that the feasible allocation (x_i) is Pareto efficient?
   (e) Let $\{p, (x_i)\}$ be an Arrow-Debreu equilibrium. Prove that (x_i) is Pareto efficient.

2. Consider an economy with I agents with utility functions (u_i). An *allocation* is denoted by (x_i), where x_i is the consumption bundle of agent i. The initial endowment is (e_i).
   (a) What does it mean that the allocation (x_i) is Pareto efficient?
   (b) What does the first welfare theorem say?
   (c) How are the assumptions of the first and second welfare theorems different?

3. Consider an economy with two agents and two goods. The utility functions are

$$\begin{aligned} u_1(x_1, x_2) &= \sqrt{x_1 x_2}, \\ u_2(x_1, x_2) &= \min\left\{\sqrt{x_1 x_2}, 4\right\}. \end{aligned}$$

   The initial endowments are $e_1 = (1, 9)$ and $e_2 = (9, 1)$.
   (a) Show that the price vector $(p_1, p_2) = (1, 1)$ and the allocation $x_1 = x_2 = (5, 5)$ constitute a competitive equilibrium.
   (b) Show that the equilibrium allocation is Pareto inefficient.
   (c) Does this example contradict the first welfare theorem? Answer yes or no, and explain why.

4. Consider an economy with two agent with Cobb-Douglas utility functions

$$u_1(x_1, x_2) = \frac{2}{3}\log x_1 + \frac{1}{3}\log x_2,$$
$$u_2(x_1, x_2) = \frac{1}{3}\log x_1 + \frac{2}{3}\log x_2.$$

Assume that the endowments are $e_1 = (1, 2)$ and $e_2 = (1, 3)$.

(a) Consider the allocation (x_1, x_2) (for agent 1) and $(2 - x_1, 5 - x_2)$ (for agent 2). Compute the marginal rate of substitution

$$\frac{\partial u_i}{\partial x_2} / \frac{\partial u_i}{\partial x_1}$$

for each agent $i = 1, 2$ at this allocation.

(b) Show that the initial endowment $\{(1, 2), (1, 3)\}$ is Pareto inefficient.

(c) Show that the allocation $\{(1, 1), (1, 4)\}$ is Pareto efficient.

(d) Compute the price vector $p = (p_1, p_2)$ (with $p_1 = 1$) and transfer payments (t_1, t_2) such that the price p and the allocation $\{(1, 1), (1, 4)\}$ constitute a competitive equilibrium with transfer payments.

5. Consider an economy with two agents indexed by $i = 1, 2$ and two goods indexed by $l = 1, 2$. The utility functions are

$$u_1(x_1, x_2) = -\frac{1}{x_1} - \frac{1}{x_2},$$
$$u_2(x_1, x_2) = \frac{2}{3}\log x_1 + \frac{1}{3}\log x_2,$$

and the initial endowments are $e_1 = e_2 = (1, 6)$.

(a) Is the initial endowment Pareto efficient? Answer yes or no, then explain why.

(b) Compute the Pareto efficient allocation in which agent 1 consumes 1 unit of good 1.

(c) Compute the competitive equilibrium with transfer payments when the allocation is the one in the previous question. (Normalize the price of good 1 to be 1, so $p_1 = 1$.)

4 Existence

1. (a) Define $\Gamma : \mathbb{R} \twoheadrightarrow \mathbb{R}$ by

$$\Gamma(x) = \begin{cases} \{0\}, & (x < 0) \\ [0, 1]. & (x \geq 0) \end{cases}$$

Show that Γ is upper hemicontinuous but not lower hemicontinuous.

(b) Define $\Gamma : \mathbb{R} \twoheadrightarrow \mathbb{R}$ by

$$\Gamma(x) = \begin{cases} \{0\}, & (x \leq 0) \\ [0, 1]. & (x > 0) \end{cases}$$

Show that Γ is lower hemicontinuous but not upper hemicontinuous.

2. To see why the assumption $e_i \gg 0$ is necessary for the existence of equilibrium, show that the budget set is not lower hemicontinuous when $L = 2$, $e_i = (1, 0)$, and $p = (0, 1)$.

3. Instead of assuming that any bundle $x \in \mathbb{R}^L_+$ is physically possible, let $X_i \subset \mathbb{R}^L_+$ be the consumption set (the set of all physically possible consumption bundles) of agent i. For example, if good 1 is "leisure" (time measured in hours) and other goods can be consumed in any amount, then $X_i = [0, 24] \times \mathbb{R}^{L-1}_+$.

   Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. Prove that an equilibrium exists if for all i (i) X_i is closed and convex, (ii) $e_i \in X_i$, and there exists a point $\bar{x}_i \in X_i$ such that $e_i \gg \bar{x}_i$, and (iii) $u_i : X_i \to \mathbb{R}$ is continuous, quasi-concave, and locally nonsatiated.

4. Let $\mathcal{E} = \{I, J, (e_i), (u_i), (Y_j), (\theta_{ij})\}$ be an Arrow-Debreu economy with production. Assume that for all j, $0 \in Y_j$ (possibility of inaction) and Y_j is compact and convex. Prove the existence of equilibrium under suitable assumptions.

5. We say that there is an *externality* when the utility of some agent depends on the consumption of some other agent, such as a neighbor's ugly yard full of weeds (negative externality) or another neighbor's beautiful yard full of flowers (positive externality). Consider an economy $\mathcal{E} = \{I, (e_i), (u_i)\}$, where $u_i(x)$ is the utility of agent i when the allocation is $x = (x_i) = (x_1, \dots, x_I) = (x_i, x_{-i}) \in \mathbb{R}^{LI}_+$. Here $x_i \in \mathbb{R}^L_+$ denotes the consumption of agent i and $x_{-i} \in \mathbb{R}^{L(I-1)}_+$ denotes the consumption bundles of all other agents.

   Assume that for all i, $e_i \gg 0$, $u_i(x)$ is continuous in x, and $u_i(x_i, x_{-i})$ is quasi-concave and locally nonsatiated in x_i. Verify that the existence of a competitive equilibrium can be proved in the exact same manner.

5 International trade

1. Consider an economy with three goods and two agents, all with the utility function

   $$u(x_1, x_2, x_3) = \frac{1}{3}\log x_1 + \frac{1}{3}\log x_2 + \frac{1}{3}\log x_3.$$

   Labor endowments are $(e_A, e_B) = (1, 3)$ and productivities are

   $$\begin{aligned}(a_{A1}, a_{A2}, a_{A3}) &= (12, 9, 8),\\ (a_{B1}, a_{B2}, a_{B3}) &= (6, 4, 3).\end{aligned}$$

   Compute the equilibrium in autarky for each country and the free trade equilibrium.

2. Consider an economy with two countries, $i = A, B$, and three consumption goods, $l = 1, 2, 3$. Both countries have labor endowment $e_A = e_B = 3$. The utility functions are

   $$\begin{aligned}u_A(x_1, x_2, x_3) &= \frac{1}{2}\log x_1 + \frac{1}{4}\log x_2 + \frac{1}{4}\log x_3,\\ u_B(x_1, x_2, x_3) &= \frac{1}{3}\log x_1 + \frac{1}{3}\log x_2 + \frac{1}{3}\log x_3.\end{aligned}$$

Each country can produce the consumption goods from labor using the linear technology $y = a_{il}e$, where e is labor input, y is output of good l, and $a_{il} > 0$ is the productivity. Assume that productivities are

$$(a_{A1}, a_{A2}, a_{A3}) = (4, 2, 2),$$
$$(a_{B1}, a_{B2}, a_{B3}) = (1, 1, 2).$$

(a) What is the definition of comparative advantage of country A over B? Compute the comparative advantage for each industry.

(b) Given the price $p = (p_1, p_2, p_3)$ and the wage w_A of country A, compute the demand of country A.

(c) Assuming that both countries produce good 2 in free trade and setting $p_2 = 1$, compute p_1, p_3, w_A, w_B.

(d) Compute the free trade equilibrium consumption in each country.

(e) Compute the labor allocation across each industry for each country.

3. Consider a world with L goods indexed by $l = 1, \dots, L$. Let $p = (p_1, \dots, p_L)$ be the vector of world prices. Suppose that a small country (hence it does not affect world price p) has I citizens indexed by $i = 1, \dots, I$, and let $u_i(x)$ be the (locally nonsatiated) utility function of agent i and e_i be the initial endowment.

(a) Suppose that the government is adopting some trade policy (tariff, quotas, etc.), and the equilibrium allocation is (x_i). If the government is neither running a trade surplus nor a deficit, show that it must be

$$\sum_{i=1}^{I} p \cdot (e_i - x_i) = 0.$$

(b) Find a trade policy that weakly Pareto improves the initial trade policy. Explain why your policy is weakly Pareto improving.

4. Consider an economy with three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), and two countries, A, B. Agents 1 and 2 live in country A and agent 3 lives in country B. The utility functions are

$$u_1(x_1, x_2) = x_1^2 x_2,$$
$$u_2(x_1, x_2) = x_1 x_2^2,$$
$$u_3(x_1, x_2) = x_1 x_2.$$

Suppose that the initial endowments are $e_1 = e_2 = (3, 3)$ and $e_3 = (22, 8)$.

(a) Compute the competitive equilibrium when country A is in autarky as well as the utility level of each agent.

(b) Compute the free trade equilibrium price and allocation.

(c) Compute the utility level of each agent and determine who gained from trade and who lost.

(d) Find a tax scheme in country A such that free trade is Pareto improving. Explain why the tax scheme you suggest is Pareto improving.

5. Consider an economy with two countries, $i = A, B$, and two physical goods, $l = 1, 2$. The endowment is $e_A = (6, 1)$ and $e_B = (1, 6)$. The utility function is $u(x_1, x_2) = x_1 x_2$ for all agents. Suppose that there are transportation costs, and 50% of the exported goods perish by the time they reach the destination.
   (a) How many kinds of goods are there in the world? Answer the number and explain the reason.
   (b) Assuming that country A imports good 2, what is its price? (Set the price of good 1 equal to 1.)
   (c) Compute the free trade equilibrium.

6. Consider an economy with two goods indexed by $l = 1, 2$. Suppose that there is a small country (so it doesn't affect world prices) with two agents indexed by $i = 1, 2$ and endowments $e_1 = (3/2, 1)$, $e_2 = (1, 3/2)$. All agents have utility function

$$u(x_1, x_2) = x_1 x_2.$$

   Below, always normalize the price of good 1 to be $p_1 = 1$.
   (a) Compute the autarky equilibrium allocation and price.
   (b) Suppose that the country opens up to trade, and the price of good 2 changes to $p_2 = 1/2$. Compute the free trade allocation and utility and determine who gains/loses from trade.
   (c) Suppose that the government imposes a tariff of $\tau = 1/4$ on the import of good 2, and the domestic price of good 2 becomes $q = p_2 + \tau = 1/2 + 1/4 = 3/4$. Let T be the tax revenue from the tariff, and suppose that the government gives out the tariff revenue equally to agents (so each agent gets $T/2$). Derive an equation that T satisfies.
   (d) Solve for the new allocation and show that all agents gain from trade.
   (e) Propose a better policy than the government's. Compute the allocation and utility under your suggested policy.

7. Consider an economy with two countries ($i = 1, 2$) and two goods ($l = 1, 2$). Each country consists of a single agent type whose utility function is

$$u(x_1, x_2) = x_1 x_2.$$

   Suppose that the labor endowments are $e_A = 10$, $e_B = 1$, and the vector of labor productivities are

$$(a_{A1}, a_{A2}) = (12, 6)$$
$$(a_{B1}, a_{B2}) = (4, 4).$$

   (a) Compute the autarky equilibrium in country A and the utility level. Note that you need to compute prices, wage, and allocations of goods and labor. Normalize the price of good 1 so that $p_1 = 1$.
   (b) Repeat the previous question for country B.
   (c) Compute the free trade equilibrium.

(d) Explain why during international conflicts, large/developed countries often try to impose an embargo on small/developing countries. (Imagine the U.S.-Japan relationship before WWII or U.S.-North Korea now.)

8. Consider Ricardo's international trade model with two countries, $i = A, B$, and L consumption goods indexed by $l = 1, \dots, L$. Let $a_{il} > 0$ be the labor productivity of country i when producing good l. Assume that the values $\{a_{Al}/a_{Bl}\}_{l=1}^{L}$ are all distinct.

   (a) Define the notion of comparative advantage.

   (b) Let $w_i > 0$ be the wage rate in country i and $p_l > 0$ be the price of good l. In equilibrium, prove that $p_l a_{il} \leq w_i$, with equality if country i produces good l.

   (c) Let $L = \{1, \dots, L\}$ be the set of goods. Let L_i ($i = A, B$) be the set of goods produced by country i in equilibrium. Prove that $L_A \cup L_B = L$ and $L_A \cap L_B$ is either empty or consists of a single element.

9. Consider an economy with two countries ($i = 1, 2$) and two goods ($l = 1, 2$). Each country consists of a single agent type whose utility function is

$$u(x_1, x_2) = x_1 x_2.$$

   Suppose that the labor endowments are $e_A = 1$, $e_B = e$, and the vector of labor productivities are

$$(a_{A1}, a_{A2}) = (2, 1)$$
$$(a_{B1}, a_{B2}) = (1, 1).$$

   Below, normalize the price of good 1 so that $p_1 = 1$.

   (a) Define the notion of comparative advantage.

   (b) Under what condition on e is there a free trade equilibrium in which country A produces both goods? Compute the equilibrium.

   (c) Under what condition on e is there a free trade equilibrium in which country B produces both goods? Compute the equilibrium.

   (d) Under what condition on e is there a free trade equilibrium in which each country produces only one good? Compute the equilibrium.

6 Finance

1. Suppose that there are two assets, a stock and a (risk-free) bond. The current stock price is 100 and can either go up to 120 or go down to 75 tomorrow. The risk-free interest rate is 5%. In answering the questions below, always use fractions.

   (a) Let u, d stand for the up and down states and p_u, p_d be the state prices. Derive two equations that p_u, p_d satisfy.

   (b) Compute p_u, p_d.

   (c) Compute the price of a call option with strike 100.

   (d) Compute the price of a put option with strike 100.

(e) Compute the price of a convertible bond that promises to pay 100 tomorrow. (A convertible bond is a promise to pay 100, with an option to deliver the stock instead.)

2. Consider an economy with one physical good (raw fish), two periods ($t = 0, 1$), and two states at $t = 1$, denoted by $s = 1, 2$. Let π_s be the probability of state $s = 1, 2$, where $\pi_1 + \pi_2 = 1$. Suppose that agents (fishermen) have utility functions

$$u(x_0, x_1, x_2) = (1 - \beta) \log x_0 + \pi_1 \beta \log x_1 + \pi_2 \beta \log x_2,$$

where $0 < \beta < 1$ is a parameter and x_0, x_1, x_2 are consumption of fish at $t = 0$, state 1, and state 2.

(a) How many goods are there in this economy?

(b) Suppose that fishermen are sophisticated and trade future contracts that are contingent on states. For example, a state 1 contract delivers 1 fish in state 1. Let $p_0 = 1$ be the price of fish at $t = 0$ and p_s $(s = 1, 2)$ be the future price state s future contract. If an agent has wealth w, what is his demand?

(c) Let e_0, e_1, e_2 be the aggregate endowment of fish and $W = e_0 + p_1 e_1 + p_2 e_2$ be the aggregate wealth. Show that

$$\frac{1 - \beta}{e_0} = \frac{\pi_1 \beta}{p_1 e_1} = \frac{\pi_2 \beta}{p_2 e_2} = \frac{1}{W}.$$

(d) Compute the future prices p_1, p_2.

(e) Consider an asset that pays out the aggregate endowment of fish at $t = 1$ as dividend. Compute the asset price.

(f) Compute the gross risk-free rate in this economy.

3. Consider an economy with two periods, denoted by $t = 0, 1$, and three agents, denoted by $i = 1, 2, 3$. There are two states at $t = 1$, denoted by $s = 1, 2$. The two states occur with equal probability $\pi_1 = \pi_2 = 1/2$. Suppose that agent i's utility function is

$$U_i(x_0, x_1, x_2) = u_i(x_0) + \pi_1 u_i(x_1) + \pi_2 u_i(x_2),$$

where x_0, x_1, x_2 denote the consumption at $t = 0$ and states $s = 1, 2$, and the Bernoulli utility functions $u_i(x)$ are given by

$$\begin{aligned} u_1(x) &= \sqrt{2x}, \\ u_2(x) &= \sqrt{2x - 2}, \\ u_3(x) &= \sqrt{2x + 2}. \end{aligned}$$

The initial endowments $e_i = (e_{i0}, e_{i1}, e_{i2})$ are given by

$$\begin{aligned} e_1 &= (1, 2, 5), \\ e_2 &= (2, 2, 4), \\ e_3 &= (3/2, 4, 7/2). \end{aligned}$$

(a) What is the name of this type of utility functions?

(b) For a given level of consumption, which agent is the most risk averse? Answer based on reasoning.

(c) Normalize the price of $t = 0$ good to be $p_0 = 1$. Compute the equilibrium state prices p_1, p_2.

(d) Compute the (gross) risk-free interest rate.

(e) Consider an asset (stock) that pays out the aggregate endowment as dividend. Compute the ex-dividend stock price (the stock price excluding the dividend) at $t = 0$.

(f) Compute the expected stock return at $t = 0$ and show that it is higher than the risk-free rate.

(g) Compute the price at $t = 0$ of a call option written on a stock with strike price 10.

4. What is the Mutual Fund Theorem? Explain the statement, assumptions, and practical implications.

5. Consider an economy with two periods, denoted by $t = 0, 1$, and three agents, denoted by $i = 1, 2, 3$. There are two states at $t = 1$, denoted by $s = 1, 2$. The two states occur with equal probability $\pi_1 = \pi_2 = 1/2$. Suppose that agent i's utility function is

$$U_i(x_0, x_1, x_2) = u_i(x_0) + \pi_1 u_i(x_1) + \pi_2 u_i(x_2),$$

where x_0, x_1, x_2 denote the consumption at $t = 0$ and states $s = 1, 2$, and the Bernoulli utility functions $u_i(x)$ are given by

$$\begin{aligned} u_1(x) &= -\frac{4}{x}, \\ u_2(x) &= -\frac{4}{x-2}, \\ u_3(x) &= -\frac{4}{x+2}. \end{aligned}$$

The initial endowments $e_i = (e_{i0}, e_{i1}, e_{i2})$ are given by

$$\begin{aligned} e_1 &= (3, 5, 1), \\ e_2 &= (3, 3, 3), \\ e_3 &= (2, 4, 2). \end{aligned}$$

(a) What is the name of this type of utility functions?

(b) For a given level of consumption, which agent is the most risk averse? Answer based on reasoning.

(c) Normalize the price of $t = 0$ good to be $p_0 = 1$. Compute the equilibrium state prices p_1, p_2.

(d) Compute the (gross) risk-free interest rate.

(e) Consider an asset (stock) that pays out the aggregate endowment as dividend. Compute the ex-dividend stock price (the stock price excluding the dividend) at $t = 0$.

(f) Compute the expected stock return at $t = 0$ and show that it is higher than the risk-free rate.

(g) Compute the price at $t = 0$ of a put option written on a stock with strike price 9.

6. Consider the binomial option pricing model discussed in the lectures. Time is denoted by $t = 0, 1, \dots, T$. The gross risk-free rate is constant at $R > 0$. Each period, the stock can go up or down, so

$$S_{t+1} = \begin{cases} US_t & \text{if stock goes up,} \\ DS_t & \text{if stock goes down,} \end{cases}$$

where $U > R > D$. Suppose the initial stock price $S_0 > 0$ is given and consider an American put option with strike price K and expiration T. Recall that a put option is the right to sell a stock at the strike price, and "American" means that the option can be exercised any time until expiration.

(a) Suppose for simplicity that $T = 1$. Let u, d stand for the up and down states and p_u, p_d be the state prices. Derive two equations that p_u, p_d satisfy.

(b) Compute p_u, p_d.

(c) Compute the price of the American put option.

(d) What is the relation between the interest rate R and the put option price? Is put price increasing, decreasing, or ambiguous in R?

(e) How does your answer to the previous question change for general expiration date T?

7. Consider an economy with two agents ($i = 1, 2$), two periods ($t = 0, 1$), and two states ($s = u, d$) at $t = 1$. The aggregate endowment is $e = (e_0, e_u, e_d)$, where $e_0 > 0$ and $e_u > e_d > 0$ (so u, d are the "up" and "down" states). The initial endowments are $e_1 = \alpha e$ and $e_2 = (1-\alpha)e$, so the wealth share of agent 1 is $0 < \alpha < 1$. Agent 1's subjective probability of state s is $\pi_s > 0$, where $\pi_u + \pi_d = 1$. The utility functions are

$$\begin{aligned} u_1(x_0, x_u, x_d) &= (1-\beta)\log x_0 + \beta(\pi_u \log x_u + \pi_d \log x_d), \\ u_2(x_0, x_u, x_d) &= (1-\beta)\log x_0 + \beta \log(\min\{x_u, x_d\}), \end{aligned}$$

where $0 < \beta < 1$ is the discount factor. Let $p = (1, p_u, p_d)$ be the price vector and $W = e_0 + p_u e_u + p_d e_d$ be the value of aggregate endowment. Note that this is an Arrow-Debreu economy, so all trades occur at $t = 0$.

(a) Compute the demand of agent 1 using p_u, p_d, W, and exogenous parameters.

(b) Compute the demand of agent 2 using p_u, p_d, W, and exogenous parameters.

(c) Let a "stock" be an asset that pays out the aggregate endowment at $t = 1$. Let a "risk-free asset" be an asset that pays out 1 at $t = 1$ no matter what. Let q_s, q_f be the price of the stock and risk-free asset at $t = 0$. Express q_s, q_f using only p_u, p_d, e_u, e_d.

(d) Express p_u, p_d using only q_s, q_f, e_u, e_d.

(e) Using the market clearing condition for $t = 0$, compute q_s using only exogenous parameters.

(f) Using the market clearing condition for $s = u, d$, show that

$$\frac{\alpha\beta}{1-\beta} e_0 \left(\frac{\pi_u}{p_u} - \frac{\pi_d}{p_d} \right) = e_u - e_d.$$

(g) Show that the risk-free rate goes up when agent 1 becomes relatively richer (i.e., α gets larger).

8. Consider the binomial option pricing model discussed in the lectures. Time is denoted by $t = 0, 1, \dots, T$. The gross risk-free rate is constant at $R > 0$. Each period, the stock can go up or down, so

$$S_{t+1} = \begin{cases} US_t & \text{if stock goes up,} \\ DS_t & \text{if stock goes down,} \end{cases}$$

where $U > R > D$. Suppose the initial stock price $S_0 > 0$ is given and consider an American put option with strike price K and expiration T. Recall that a put option is the right to sell a stock at the strike price, and "American" means that the option can be exercised any time until expiration.

(a) Suppose for simplicity that $T = 1$. Let u, d stand for the up and down states and p_u, p_d be the state prices. Derive two equations that p_u, p_d satisfy.

(b) Compute p_u, p_d.

(c) Compute the price of the American put option.

(d) Now suppose that the expiration date T is arbitrary. Prove that the put option price is decreasing in the current stock price S_0.

(e) Prove that the put option price is convex in the current stock price S_0.

7 Miscellaneous

1. A course (not this course) has two midterms and a final exam. The raw scores are denoted by M_1, M_2, F, which can take continuous values between 0 and 100. Since the raw scores were somewhat disappointing, the (crazy) professor decided to replace them by

$$\begin{aligned} M_1' &= \frac{5}{6}\sqrt{M_1 + 41}, \\ M_2' &= \frac{1}{2}\max\left\{ M_1 - 7, 2M_2 - \frac{1}{100}M_2^2 \right\} + \frac{1}{3}M_2 + \frac{1}{6}M_1, \\ F' &= 10 + 4\sqrt{\max\{M_2, F\}} + \frac{1}{3}F + \frac{1}{6}\min\{M_1, F\}. \end{aligned}$$

Prove that there exist raw scores (M_1, M_2, F) that remain unchanged under this transformation.

2. Consider an economy with two periods denoted by $t = 0, 1$ (today and tomorrow). There is a single good at each period. As explained in the lecture, goods are distinguished by time, so introducing time does not change the model (at least formally). The good is perishable, like raw fish, so the good today cannot be transformed into the good tomorrow.

   (a) Let p_0, p_1 be the price of 1 unit of good at each period, so actually p_1 is the *future price* (the amount of money you have to pay today in order to be delivered 1 unit of good tomorrow). Let the interest rate be $100r\%$. By arguing how much good you will get tomorrow by reducing the consumption of 1 unit of good today, express $1 + r$ as a function of p_0 and p_1.

   (b) Suppose that there are two agents, A and B, and the endowment of each agent is $e^A = e^B = (e_0, e_1)$. (So A and B go fishing and fish e_0 at $t = 0$ and e_1 at $t = 1$.) Each agent has the utility function

   $$u^A(x_0, x_1) = (1 - \beta^A) \log x_0 + \beta^A \log x_1,$$
   $$u^B(x_0, x_1) = (1 - \beta^B) \log x_0 + \beta^B \log x_1.$$

   Let the price be $p_0 = 1$ and $p_1 = p$. Compute the demand of agent A, given the price p.

   (c) Compute the equilibrium price and interest rate.

   (d) Assume $\beta^A > \beta^B$. Which agent is saving? Which agent is borrowing?

3. Consider an economy with two goods and two agents. The utility functions are

   $$U_1(x_1, x_2) = x_1 - \frac{1}{2x_2^2},$$
   $$U_2(x_1, x_2) = -\frac{1}{2x_1^2} + x_2.$$

   The endowments are $e_1 = e_2 = (e, e)$, where $e > 0$. Assume that agent 1 can consume good 1 in negative amounts, and agent 2 can consume good 2 in negative amounts.

   (a) Let the prices be $p_1 = 1$ and $p_2 = p$. Compute the demand of agent 1.

   (b) Let $z_1(p)$ be the aggregate excess demand of good 1. Compute $z_1(p)$.

   (c) Show that $z_1(1) = 0$ and $z_1(\infty) = \infty$.

   (d) Compute $z_1'(1)$.

   (e) Show that this economy has more than one equilibria if $0 < e < \frac{1}{3}$.

4. Consider an economy with I agents and L basic goods labeled by $l = 1, \ldots, L$. Suppose that there is another good, labeled 0, which is a public good. (A public good is non-excludable, i.e., the consumption of one agent does not reduce the availability of that good to other agents. Therefore all agents consume the same amount of good 0, which equals aggregate supply in equilibrium.) Suppose that there are no endowments of good 0, which is produced from other goods $l = 1, \ldots, L$ using some technology represented by a production function $y = f(x_1, \ldots, x_L)$. It is well known that the presence of a public good may make the economy inefficient. Let $u_i(x_0, x_1, \ldots, x_L)$ be the utility function of agent i, assumed to be locally nonsatiated.

(a) Show that by quoting an individual-specific price for the public good, we can make the competitive equilibrium allocation efficient. (Hint: expand the set of goods, and consider an economy with $L+I$ goods labeled by $l = 1, \dots, L+I$. Goods $l = 1, \dots, L$ are the L basic goods, and good $L+i$ is the public good consumed by agent i. Make sure to discuss how we should reinterpret the production technology and market clearing conditions.)

(b) Suppose that there are two agents ($i = 1, 2$), one basic good, and a public good. Agent i has utility function

$$u_i(x_0, x_1) = \alpha_i \log x_0 + (1 - \alpha_i) \log x_1,$$

where x_0, x_1 are consumption of the public good and the basic good. Let e_i be the initial endowment of agent i's basic good, and suppose there is a technology that converts the basic good to the public good one-for-one. Normalize the price of the basic good to be 1. Find individual-specific prices for the public good to make the competitive equilibrium allocation efficient.

5. Consider an economy consisting of two agents, $i = A, B$, and two goods, $l = 1, 2$. The initial endowments are $e_A = a = (a_1, a_2) \gg 0$, $e_B = b = (b_1, b_2) \gg 0$, and agents have Cobb-Douglas utility functions

$$U_A(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2,$$
$$U_B(x_1, x_2) = \beta \log x_1 + (1 - \beta) \log x_2,$$

where $\alpha, \beta \in (0, 1)$. Below, normalize the price of good 2 to be $p_2 = 1$ and set $p_1 = p$.

(a) Let $w = pa_1 + a_2$ be the wealth of agent A. Write down the Lagrangian of the utility maximization problem.

(b) Derive the first-order conditions.

(c) Express the demand of agent A using only α, a_1, a_2, p.

(d) Compute the equilibrium price.

(e) Show that if either (i) the supply of good 1 decreases (a_1 or b_1 goes down), (ii) the supply of good 2 increases (a_2 or b_2 goes up), or (iii) agents like good 1 more (α or β goes up), then the price of good 1 increases.

6. (a) Let $f(x) = -\frac{1}{\gamma} e^{-\gamma x}$, where $\gamma > 0$. Compute $-\frac{f''(x)}{f'(x)}$.

(b) Consider an economy with two goods that can be consumed in any amounts (positive or negative), and suppose that an agent has endowment $e = (a, b)$ and an additively separable utility function

$$U(x, y) = u(x) + v(y),$$

where $u' > 0$, $u'' < 0$, and similarly for v. Let $p_1 = p$ and $p_2 = 1$ be prices. Show that the agent's demand (x, y) satisfies

$$u'(x) - pv'(pa + b - px) = 0.$$

(c) Regard x, the demand for good 1, as a function of the price p. Show that

$$\frac{\partial x}{\partial p} = \frac{v'(y) + pv''(y)(a-x)}{u''(x) + p^2 v''(y)},$$

where $y = pa + b - px$.

(d) Show that

$$\frac{\partial x}{\partial p} = -\frac{1 - p\gamma_v(y)(a-x)}{p\gamma_u(x) + p^2\gamma_v(y)},$$

where $\gamma_u(x) = -\frac{u''(x)}{u'(x)}$ and similarly for v.

(e) Suppose that there are I agents indexed by $i = 1, \dots, I$. Agent i's endowment is $e_i = (a_i, b_i)$ and the utility function is

$$U_i(x, y) = -\frac{1}{\gamma_i}\left(\alpha_i \mathrm{e}^{-\gamma_i x} + (1-\alpha_i)\mathrm{e}^{-\gamma_i y}\right),$$

where $\gamma_i > 0$ and $0 < \alpha_i < 1$. Let x_i be agent i's demand for good 1. Show that

$$\frac{\partial x_i}{\partial p} = -\frac{1}{\gamma_i p(1+p)} + \frac{1}{1+p}(a_i - x_i).$$

(f) Let $z_1(p) = \sum_{i=1}^{I}(x_i - a_i)$ be the aggregate excess demand for good 1. Show that if p is an equilibrium price, then $z_1'(p) < 0$.

(g) Show that the equilibrium is unique.

7. Consider an Arrow-Debreu economy with two agents denoted by $i = A, B$ and S states denoted by $s = 1, \dots, S$. Let $\pi_s > 0$ be the objective probability of state s, where $\sum_{s=1}^{S} \pi_s = 1$. Let $e_{is} > 0$ be agent i's initial endowment of good s. Suppose that the utility functions are given by

$$U_A(x) = \sum_{s=1}^{S} \pi_s u(x_s),$$
$$U_B(x) = \sum_{s=1}^{S} \pi_s x_s,$$

where $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. What is the most you can say about equilibrium prices and allocations?

8. Consider an Arrow-Debreu economy with two agents indexed by $i = 1, 2$. Suppose that the utility functions are

$$U_1(x_1, x_2) = \alpha \log x_1 + (1-\alpha)\log x_2,$$
$$U_2(x_1, x_2) = \min\{x_1, x_2\},$$

where $0 < \alpha < 1$ is a preference parameter. Let agent i's initial endowment be (e_{i1}, e_{i2}). Let $p_1 = 1$ and $p_2 = p$ be the prices.

(a) Compute each agent's demand for good 1, given p.

(b) Derive a necessary and sufficient condition such that p is an equilibrium price for some $e_{12} > 0$.

(c) Show that when agent 1 likes good 1 more (α increases), the relative price of good 1, which is $1/p$, increases (so p decreases).

9. Consider the following general equilibrium model. There are three time periods indexed by $t = 0, 1, 2$. There is a continuum of ex ante identical agents, where the population is normalized to 1. At $t = 0$, agents are endowed with $e > 0$ units of consumption good. At $t = 0$, agents can invest goods in two technologies. One unit of investment in technology 1 yields 1 unit of good at $t = 1$. One unit of investment in technology 2 yields $R > 0$ units of good at $t = 2$. Agents get utility only from consumption at $t = 1, 2$. At the beginning of $t = 1$, agents get "liquidity shocks", and with probability $\pi_i > 0$, their utility function becomes

$$U_i(x_1, x_2) = (1 - \beta_i) \log x_1 + \beta_i \log x_2,$$

where $\beta_i \in (0, 1)$ is the discount factor of type i and $\sum_{i=1}^{I} \pi_i = 1$. Without loss of generality, assume

$$\beta_1 < \cdots < \beta_I,$$

so a type with a smaller index is more impatient. Suppose that the ex ante utility is

$$U((x_{i1}, x_{is})_i) = \sum_{i=1}^{I} \pi_i \alpha_i U_i(x_{i1}, x_{i2}),$$

where $\alpha_i > 0$ is the weight on type i such that

$$\alpha_1 > \cdots > \alpha_I,$$

so agents care about emergencies in the sense that they put more utility weight on the impatient type. Note that we assume the law of large numbers, so at $t = 1$, exactly fraction $\pi_i > 0$ of agents are of type i. After observing their patience type at $t = 1$, agents can trade consumption for $t = 1, 2$ at a competitive (Arrow-Debreu) market.

(a) In general, let f, g be strictly increasing functions and X be a random variable. Prove the Chebyshev inequality

$$E[f(X)g(X)] \geq E[f(X)]E[g(X)],$$

with equality if and only if X is constant almost surely.

(Hint: let X' be an i.i.d. copy of X and consider the expectation of the quantity $(f(X) - f(X'))(g(X) - g(X')) \geq 0$.)

(b) Noting that agents are ex ante identical, at $t = 0$ they will all make the same investment decision. let $x \in (0, e)$ be the amount of investment in technology 1 and let $(e_1, e_2) = (x, R(e - x))$ be the vector of $t = 1, 2$ endowments conditional on x. Let $(p_1, p_2) = (1, p)$ be the price of consumption at $t = 1, 2$. Compute type i's demand for the $t = 1, 2$ goods using p, e_1, e_2.

(c) For notational simplicity, let $\bar{\beta} = \sum_{i=1}^{I} \pi_i \beta_i$ be the average discount factor. Conditional on x, compute the equilibrium price p.

(d) Let $V(x,p) = \max U((x_{i1}, x_{i2})_i)$ be the agents' maximized utility conditional on short-term investment x and price p. Noting that agents choose x optimally given p, compute the equilibrium short-term investment x^*.

(e) Suppose that the government can force the agents to choose a particular x, without interfering in the subsequent consumption markets at $t = 1, 2$. Let $p(x)$ be the price of $t = 2$ consumption conditional on x derived above. Prove that

$$\frac{d}{dx} V(x, p(x))\bigg|_{x=x^*} > 0,$$

so welfare locally increases if the government forces the agents to invest more in the short-term investment technology.

10. Consider an economy with two goods and two agents $i = A, B$. Both agents have utility function

$$u(x_1, x_2) = \min\{x_1, x_2\}.$$

The endowments are $e_A = (1, a)$ and $e_B = (1, b)$, where $a, b > 0$. Below, normalize prices such that $p_1 + p_2 = 1$.

(a) What is the name of this utility function?

(b) Does this economy necessarily have an equilibrium? Explain.

(c) If an equilibrium exists, is it necessarily Pareto efficient? Explain.

(d) Compute all possible equilibrium prices (if they exist) and construct an equilibrium allocation for each price.

(e) Suppose $a + b = 2$, $a > 1$, and $p_1 \in (0, 1)$. Show that if agent A destroys a small enough amount of endowment of good 2, he becomes strictly better off in equilibrium. Is this strange?

11. During the COVID-19 pandemic, after a brief (but severe) stock market crash in March 2020, the stock market recovered and recorded all-time highs towards the end of 2020, despite being in a recession. This problem presents a simple explanation.

Consider a general equilibrium model with two agents denoted by $i = 1, 2$ and two dates denoted by $t = 0, 1$. There is a single perishable good (apple) at each date. Agent i has utility function

$$u_i(x_0, x_1) = (1 - \beta_i) \log x_0 + \beta_i \log x_1,$$

where $\beta_i \in (0, 1)$ is agent i's discount factor. Let $e = (e_0, e_1)$ be the aggregate endowment. Suppose that agent 1's endowment is αe and agent 2's endowment is $(1 - \alpha)e$, where $\alpha \in (0, 1)$ is the wealth share of agent 1. Normalize prices such that $p_0 = 1$ and $p_1 = p$. Note that $p_0 = 1$ is the spot price of the apple, and $p_1 = p$ is the price of an apple future contract, which can be thought of as the stock market index.

(a) Let $\beta(\alpha) := \beta_1 \alpha + \beta_2 (1 - \alpha)$ be the average discount factor. Compute the equilibrium price p using e_0, e_1, and $\beta(\alpha)$.

(b) Suppose that agents 1 and 2 are the "highly educated" and "poorly educated". More specifically, agent 1 is more patient than agent 2, so $\beta_1 > \beta_2$. Furthermore, agent 1 can work remotely during a pandemic while agent 2 cannot, so a pandemic effectively transfers wealth from agent 2 to 1 (α increases). By proving that p is strictly increasing in both α and e_0, show that a stock market boom ($p \uparrow$) during a recession ($e_0 \downarrow$) is possible if the wealth share α of agent 1 exogenously increases during the recession.

12. In applications of general equilibrium to international trade, we have learned that (under some assumptions) free trade is weakly Pareto improving after appropriate taxes and subsidies. This question asks you to show that we can often make a strict Pareto improvement.

    Consider a single agent utility maximization problem. The utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ is continuously differentiable and satisfies $\nabla u \gg 0$. The initial endowment is $e \gg 0$. Suppose that the initial endowment e solves the utility maximization problem at the price vector $p > 0$.

    (a) Show that there exists $\lambda > 0$ such that $\nabla u(e) = \lambda p$ and that $p \gg 0$.

    (b) Fix a vector $d \in \mathbb{R}^L$ and consider the consumption bundle $e(t) := e + td$, where $t > 0$. Show that if $p \cdot d > 0$ and $t > 0$ is small enough, then $u(e(t)) > u(e)$.

    (c) Let $q > 0$ be any price vector that is not collinear with p. Prove that the maximum utility of the utility maximization problem under q is higher than $u(e)$.

13. In the lectures, we only considered general equilibrium models with complete markets (all goods are traded). However, in reality not all goods are traded (incomplete markets). This problem is an introduction to general equilibrium with incomplete markets (GEI).

    Suppose there is a single perishable good (apple), two dates denoted by $t = 0, 1$, and S states at $t = 1$ denoted by $s = 1, \dots, S$. At $t = 0$, there are only two markets (not $1 + S$), the spot market for apples and the asset (apple tree) market. One share of the asset has payoff vector $A = (A_1, \dots, A_S) \gg 0$ at $t = 1$. Normalize the spot price of the apple to $p_0 = 1$. Let $q > 0$ be the price of one share of the asset.

    (a) Consider an agent with utility function

    $$u(x) = (1 - \beta) \log x_0 + \beta \sum_{s=1}^{S} \pi_s \log x_s,$$

    where $x = (x_0, x_1, \dots, x_S)$ is consumption, $\beta \in (0, 1)$ is the discount factor, $\pi_s > 0$ is the subjective probability of state s, and $\sum_{s=1}^{S} \pi_s = 1$. Suppose the agent's initial endowment of apples is $e = (e_0, 0, \dots, 0)$ (endowed with only apples today) and the agent is endowed with n shares of the asset. Letting y be the asset demand, the agent's budget constraint is

    $$\begin{aligned} t = 0 : &\qquad x_0 + qy \le e_0 + qn =: w, \\ t = 1 : &\qquad x_s \le A_s y, \quad (s = 1, \dots, S). \end{aligned}$$

    Solve the utility maximization problem.

    (b) Now suppose that the economy consists of agents indexed by $i = 1, \dots, I$. Agent i has utility function

    $$u_i(x) = (1 - \beta_i) \log x_0 + \beta_i \sum_{s=1}^{S} \pi_{is} \log x_s,$$

where $\beta_i \in (0, 1)$ is the discount factor and $\pi_{is} > 0$ is the subjective probability of state s with $\sum_{s=1}^{S} \pi_{is} = 1$. Suppose that agent i's endowment of apples is $e_i = (e_{i0}, 0, \dots, 0)$ and the endowment of asset shares is $n_i > 0$. Under what conditions is the equilibrium Pareto efficient?

14. Consider the standard two-period overlapping generations (OLG) model. Time is denoted by $t = 0, 1, \dots$. At each time t, a new agent is born, who lives for two periods. The utility function of an agent born at time t is

$$U(y_t, z_{t+1}) = (1 - \beta) \log y_t + \beta \log z_{t+1},$$

where (y_t, z_{t+1}) is consumption when young and old. At $t = 0$, there is also an old agent who only cares about their consumption z_0.

For any t, the endowments are a when young and b when old. There is also an intrinsically useless asset (like piece of paper or gold), which is in unit supply, perfectly durable, and initially owned by the old at $t = 0$.

The budget constraint of an agent born at time t is therefore

$$\begin{aligned} &\text{Young:} && y_t + P_t x_t = a, \\ &\text{Old:} && z_{t+1} = b + P_{t+1} x_t, \end{aligned}$$

where (y_t, z_{t+1}) is consumption when young and old, x_t is asset holdings, and P_t is the price of the asset.

Obviously, because the initial old exit the economy and are endowed with one share of the asset, the budget constraint is

$$z_0 = b + P_0.$$

(a) Write a single budget constraint of an agent born at time t by eliminating x_t.

(b) Solve the utility maximization problem of an agent born at time t (by applying a relevant mathematical theorem and without omitting steps). Make sure to treat the cases $P_t = 0$ and $P_t > 0$ separately.

(c) Define a competitive equilibrium.

(d) From now on, let us focus on an equilibrium with $P_t > 0$ for all t (if it exists). Noting that the asset must be held by someone, and that the old exit the economy, in equilibrium it must be $x_t = 1$. Using this, derive an equation involving only P_t and P_{t+1}.

(e) Define $r_t = 1/P_t$. Show that the above equation reduces to a linear difference equation in r_t. Derive a necessary and sufficient condition on model parameters such that there exists a solution with $r_t > 0$ for all t, and hence an equilibrium with $P_t > 0$ exists.

(f) As you can see from the previous question, there can be many equilibria. Consider any two equilibria denoted by E_1 and E_2. What is the definition that E_2 Pareto dominates E_1?

(g) In this model, can you Pareto-rank equilibria?

15. Consider Tirole (1985)'s overlapping generations model discussed in class. Time is indexed by $t = 0, 1, \dots$. The initial old is endowed with $K_0 > 0$ units of capital. At each date t, a new generation is born, who lives for two periods. The young have 1 unit of labor endowment, whereas the old have no labor endowment. The utility function of generation t is Cobb-Douglas,

$$U(y_t, z_{t+1}) = (1 - \beta) \log y_t + \beta \log z_{t+1},$$

where (y_t, z_{t+1}) is consumption when young and old. There is a firm that produces outputs using the production function

$$F(K, L) = AK^\alpha L^{1-\alpha} + (1 - \delta)K,$$

where $A > 0$ is productivity, $\alpha \in (0, 1)$ is the capital share, $\delta \in [0, 1]$ is the capital depreciation rate, and (K, L) is capital and labor inputs.

(a) (5 points) Suppressing the time subscript, the firm seeks to maximize the profit

$$F(K, L) - wL - RK,$$

where $w > 0$ is the wage and $R > 0$ is the rent (gross return on capital). Solve the profit maximization problem, and using the labor market clearing condition $L = 1$, express w, R using only K.

(b) (5 points) Let R_{t+1} be the gross return on capital between time t and $t + 1$. Generation t's optimization problem is therefore

$$\begin{aligned} &\text{maximize} && (1 - \beta) \log y_t + \beta \log z_{t+1} \\ &\text{subject to} && z_{t+1} = R_{t+1}(w_t - y_t), \end{aligned}$$

where w_t is the wage at time t. Solve this optimization problem and express y_t using only w_t, R_{t+1}.

(c) (5 points) Suppose that capital is the only store of value (means of savings). Derive the difference equation that $\{K_t\}$ satisfies, and prove that for any $K_0 > 0$, there exists a unique equilibrium and that $\{K_t\}$ is convergent. Compute the steady state gross risk-free rate denoted by R_f.

(d) (5 points) We next consider the possibility of using an intrinsically useless asset (pure bubble) as a store of value. Let the supply of the bubble asset be 1, which is initially held by the old. For simplicity, we focus on the steady state, and let K, P be the capital stock and asset price. If $P > 0$, show that the gross return on capital must be $R = 1$.

(e) (5 points) Show that in a steady state with $P > 0$, it must be

$$K + P = \beta A(1 - \alpha)K^\alpha.$$

Using $R = 1$, compute the steady state K, P.

(f) (5 points) Derive a necessary and sufficient condition such that a steady state with $P > 0$ exists.

(g) (5 points) Consider parameter specifications such that a steady state with $P > 0$ exists. Show that the equilibrium in Part (c) is Pareto inefficient. (Hint: Take $T > 0$ large enough, and for each $t \geq T$, suppose the young at time t gives $\epsilon > 0$ of the good to the old at time t without changing anything else. Show that this transfer scheme is feasible and Pareto improving if $\epsilon > 0$ is small enough.