Applied General Equilibrium Theory

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Chapter I

Arrow-Debreu Model

What is general equilibrium?

- General equilibrium (GE) is antonym of partial equilibrium (PE)
- Partial equilibrium is what we learn in intermediate micro
  - Focus on one market
  - Demand and supply curve
- In reality, markets are interdependent
  - How much you consume depends on income (labor market)
  - Oil price $\uparrow \implies$ demand for SUV $\downarrow$, demand for EV $\uparrow$
- GE models economy as a whole, taking into account interaction of all markets

Commodities

- By definition, GE features multiple goods and services (commodities)
- Need to broadly interpret goods; they are distinguished not just by physical properties (e.g., apples and bananas) but
  - time: coffee beans available now or in 6 months
  - location: water available in California or Alaska
  - state: healthcare service when sick or healthy

GE is foundation of modern economics

- Depending on which feature we focus on, abstract GE model becomes specific model in each field

	Time	Location	Uncertainty
Time			
Location			
Uncertainty			

GE is foundation of modern economics

- Depending on which feature we focus on, abstract GE model becomes specific model in each field

	Time	Location	Uncertainty
Time	Macro		
Location		Trade	
Uncertainty			Finance

GE is foundation of modern economics

- Depending on which feature we focus on, abstract GE model becomes specific model in each field
- This course mostly deals with abstract models, except applications to international trade and finance
- Training of *how to think logically through models*

	Time	Location	Uncertainty
Time	Macro	International Macro	Macro-Finance
Location		Trade	International Finance
Uncertainty			Finance

Abstract Arrow-Debreu model

- From now on, consider abstract GE model in tradition of Arrow and Debreu
- Agents indexed by $i \in I = \{1, 2, \dots, I\}$
  - Convention: use upper case letter (here I) to denote both name of set and its cardinality; use lower case letter to denote generic element
  - This way we can simplify notation
  - I use i for indexing agents because it reminds us of *individual*
- Goods indexed by $l \in L = \{1, 2, \dots, L\}$
  - I use l for indexing goods because it reminds us of *label*

Some notations

- $\mathbb{R}$: set of real numbers; $\mathbb{R}^L$: set of L-dimensional vectors
- If $x = (x_1, \dots, x_L) \in \mathbb{R}^L$, we write
  - $x \geq 0$ ("x is nonnegative") if $x_l \geq 0$ for all l; the set of such vectors is denoted by $\mathbb{R}^L_+$
  - $x \gg 0$ ("x is positive") if $x_l > 0$ for all l; the set of such vectors is denoted by $\mathbb{R}^L_{++}$
- If $x, y \in \mathbb{R}^L$, we write
  - $x \geq y$ if $x_l \geq y_l$ for all l ($\iff x - y \geq 0$)
  - $x \gg y$ if $x_l > y_l$ for all l ($\iff x - y \gg 0$)
  - $x > y$ if $x \geq y$ and $x \neq y$
  - $\leq, <, \ll$ defined analogously
- Examples:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \ll \begin{bmatrix} 3 \\ 4 \end{bmatrix} < \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Consumption bundle

- Generic consumption bundle is denoted by

$$x = (x_1, \dots, x_L) \in \mathbb{R}^L_+$$

  - Note: consumption bundle belongs to nonnegative orthant $\mathbb{R}^L_+$ because we can't consume negative amounts
  - however, there are some exceptions for modeling purpose
- Specific consumption bundle of agent i denoted by

$$x_i = (x_{i1}, \dots, x_{il}, \dots, x_{iL}) \in \mathbb{R}^L_+$$

  - Note: when we use two subscripts, like x_{il}, first subscript (i) refers to agent and second (l) refers to good
  - Example: x_{12} is consumption of good 2 by agent 1; x_{21} is consumption of good 1 by agent 2
  - If confused, think concretely through 2-agent, 2-good case

Preferences, utility function

- We suppose agents' preferences are represented by utility functions
- If $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ is utility function of agent i, then

$$i \text{ prefers bundle } x \text{ to } y \iff u_i(x) > u_i(y)$$

- If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then

$$u_i(x) > u_i(y) \iff f(u_i(x)) > f(u_i(y))$$

- Hence preference ordering is the same if we use u_i or $v_i := f \circ u_i$
- Properties preserved by monotonic transformation (f) are called *ordinal*, otherwise *cardinal*; we are mostly interested in ordinal properties

Common utility functions (2-good case)

- Consider 2-good case for simplicity
  - General case: just change 2 to L
  - Below, Greek letters (α, σ, etc.) are positive parameters
- Cobb-Douglas: $u(x_1, x_2) = \alpha_1 \log x_1 + \alpha_2 \log x_2$
  - Can also use $v(x) = f(u(x)) = x_1^{\alpha_1} x_2^{\alpha_2}$ for $f(t) = \mathrm{e}^t$
- Leontief: $u(x_1, x_2) = \min\{x_1/\alpha_1, x_2/\alpha_2\}$
- Constant Elasticity of Substitution (CES): For $0 < \sigma \neq 1$, $u(x_1, x_2) = (\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma})^{\frac{1}{1-\sigma}}$
  - Can also use $v(x) = f(u(x)) = \frac{1}{1-\sigma}(\alpha_1 x_1^{1-\sigma} + \alpha_2 x_2^{1-\sigma})$ for $f(t) = \frac{t^{1-\sigma}}{1-\sigma}$ (Note $f'(t) = t^{-\sigma} > 0$, so f strictly increasing)
- Actually Cobb-Douglas and Leontief can be considered as special cases of CES (see exercise)

Arrow-Debreu model defined

Definition

An Arrow-Debreu economy $\mathcal{E} = \{I, (e_i), (u_i)\}$ consists of

- set of agents $I = \{1, 2, \dots, I\}$,
- endowments $(e_i) \subset \mathbb{R}^L_+$, and
- utility functions $u_i : \mathbb{R}^L_+ \to \mathbb{R}$

- For now, consider only *pure exchange* (or endowment) economies
  - Including production is more realistic and possible but does not add much insight at abstract level
- Just to remind you, agent i's endowment is a vector

$$e_i = (e_{i1}, \dots, e_{iL}) \in \mathbb{R}^L_+$$

Markets, prices

- We suppose all goods are traded (*complete market*)
- Clearly unrealistic but useful for abstraction
  - You can't sell your future labor income because you may run away without working
  - Many specific goods not traded because trade volume too small (umbrella when it rains in San Diego on specific day)
- Suppose good l quoted at price $p_l \in \mathbb{R}$ in some unit of account; price vector is $p = (p_1, \dots, p_L) \in \mathbb{R}^L$
- Normally, $p_l > 0$ (need to pay something to get good)
- But it is possible that
  - $p_l = 0$ (free good): e.g., air, swimming at beach
  - $p_l < 0$ ("bad"): e.g., garbage, junk car, nuclear waste

Budget constraint

- Recall agent i endowed with $e_i = (e_{i1}, \dots, e_{iL}) \in \mathbb{R}^L_+$
- If sell off entire endowment, receives

$$w_i := \sum_{l=1}^{L} p_l e_{il} = p \cdot e_i$$

  of unit of account (money), where $\cdot$ denotes inner product
- If agent i wishes to consume bundle $x = (x_1, \dots, x_L)$, required expenditure is

$$\sum_{l=1}^{L} p_l x_l = p \cdot x$$

- Hence budget constraint is

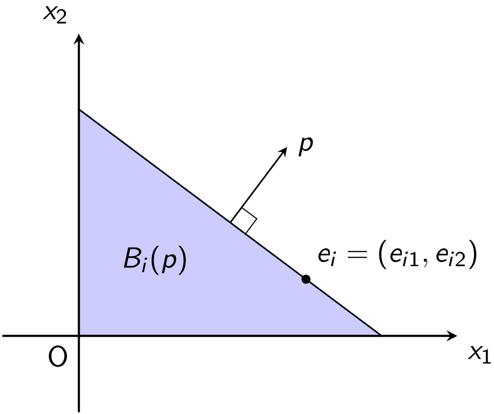
$$p \cdot x \le p \cdot e_i \iff p \cdot (x - e_i) \le 0$$

  (Inequality because not forced to spend everything)

Budget set

- Budget set:

$$B_i(p) := \left\{ x \in \mathbb{R}^L_+ \;\middle|\; p \cdot (x - e_i) \leq 0 \right\}$$



Neutrality of money

- If $t > 0$, then

$$\begin{aligned} B_i(tp) &= \left\{ x \in \mathbb{R}^L_+ \,\middle|\, (tp) \cdot (x - e_i) \le 0 \right\} \\ &= \left\{ x \in \mathbb{R}^L_+ \,\middle|\, t(p \cdot (x - e_i)) \le 0 \right\} \\ &= \left\{ x \in \mathbb{R}^L_+ \,\middle|\, p \cdot (x - e_i) \le 0 \right\} = B_i(p), \end{aligned}$$

  so scaling up or down all prices by same factor doesn't affect budget set
- Neutrality of money: it doesn't matter in which unit we quote price
- Hence we can normalize prices, for example $\sum_{l=1}^L p_l = 1$ or $p_1 = 1$

Objective of agents

- Objective of each agent is to maximize utility
- Solve

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && x \in B_i(p) \end{aligned}$$

- Suppose x_i solves utility maximization problem
- Then it also solves

$$\begin{aligned} &\text{maximize} && f(u_i(x)) \\ &\text{subject to} && x \in B_i(p), \end{aligned}$$

  where f is any strictly increasing function
- Hence optimal behavior (x_i) is ordinal property, whereas maximum utility $u_i(x_i)$ is cardinal property

Walrasian equilibrium

Definition

A competitive equilibrium (also known as Walrasian equilibrium) $\left\{p, (x_i)_{i=1}^I\right\}$ consists of a price vector $p \in \mathbb{R}^L$ and an allocation $(x_i) \subset \mathbb{R}_+^L$ such that

1. (Agent optimization) for each i, x_i solves the utility maximization problem, that is, $x_i \in B_i(p)$ and

$$x \in B_i(p) \implies u_i(x_i) \geq u_i(x),$$

2. (Market clearing) the allocation is feasible, that is,

$$\underbrace{\sum_{i=1}^I x_i}_{\text{aggregate demand}} \leq \underbrace{\sum_{i=1}^I e_i}_{\text{aggregate supply}}$$

Comments on equilibrium

- Price-taking behavior: no monopoly or monopsony
- When choosing demand, agents care only about prices and their wealth/preference; it could be that at particular price, an agent demands more goods than exist on earth
- Equilibrium is a situation in which collective behavior is consistent with aggregate resources
- Condition "demand $\le$ supply" (not demand $=$ supply) important: free disposal (unconsumed goods may be left unconsumed, like air)
  - Useful abstraction but clearly unrealistic—in reality you can't get rid of garbage, junk car, or nuclear waste for free

Example: Edgeworth box economy

- Two agents ($I = 2$), two goods ($L = 2$)
- Suppose endowments are

$$e_1 = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = e_2 = \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

- Suppose utility functions are Cobb-Douglas:

$$\begin{aligned} u_1(x_1, x_2) &= \frac{2}{3} \log x_1 + \frac{1}{3} \log x_2, \\ u_2(x_1, x_2) &= \frac{1}{3} \log x_1 + \frac{2}{3} \log x_2, \end{aligned}$$

- Let $p = (p_1, p_2)$ be price vector; then initial wealth $w_1 = w_2 = 3p_1 + 3p_2$
- What is equilibrium?

First, compute demand

- Consider utility maximization problem

$$\begin{aligned} &\text{maximize} && \alpha \log x_1 + (1-\alpha)\log x_2 \\ &\text{subject to} && p_1 x_1 + p_2 x_2 \le w \end{aligned}$$

- Lagrangian (more on this next lecture) is

$$L(x_1, x_2, \lambda) = \alpha \log x_1 + (1-\alpha)\log x_2 + \lambda(w - p_1 x_1 - p_2 x_2)$$

- How to remember: if solve

$$\begin{aligned} &\text{maximize} && f(x) \\ &\text{subject to} && g(x) \ge 0, \end{aligned}$$

  then Lagrangian is $L(x, \lambda) = f(x) + \lambda g(x)$
- In above example, $f(x) = \alpha \log x_1 + (1-\alpha)\log x_2$, $g(x) = w - p_1 x_1 - p_2 x_2$

Karush-Kuhn-Tucker conditions

- First-order condition

$$0 = \frac{\partial L}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 \implies x_1 = \frac{\alpha}{\lambda p_1},$$
$$0 = \frac{\partial L}{\partial x_2} = \frac{1-\alpha}{x_2} - \lambda p_2 \implies x_2 = \frac{1-\alpha}{\lambda p_2}$$

- Complementary slackness condition

$$0 = \lambda(w - p_1 x_1 - p_2 x_2) \iff \lambda = \frac{1}{w}$$

- Hence solution is (Cobb-Douglas formula)

$$(x_1, x_2) = \left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right)$$

- I will cover how to solve constrained optimization problems in a few lectures; for rigorous theory see graduate course material (`https://alexisakira.github.io/teaching/mathcamp`)

Going back to original problem,

- Cobb-Douglas formula: demand $= \left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right)$
- Agent 1: $\alpha = 2/3$, $w_1 = 3p_1 + 3p_2$, so

$$x_1 = (x_{11}, x_{12}) = \left(\frac{2p_1 + 2p_2}{p_1}, \frac{p_1 + p_2}{p_2}\right)$$

- Agent 2: $\alpha = 1/3$, $w_1 = 3p_1 + 3p_2$, so

$$x_2 = (x_{21}, x_{22}) = \left(\frac{p_1 + p_2}{p_1}, \frac{2p_1 + 2p_2}{p_2}\right)$$

- Can normalize price, so set $(p_1, p_2) = (1, p)$; then

$$x_1 = \left(2 + 2p, \frac{1+p}{p}\right), \quad x_2 = \left(1 + p, \frac{2+2p}{p}\right)$$

Market clearing condition

- Recall market clearing condition is "demand $\leq$ supply"
- Under weak conditions, can change to "demand $=$ supply" (more details in a few lectures)
- Hence market clearing condition for good 1 is

$$\begin{aligned} x_{11}+x_{21}=e_{11}+e_{21} &\iff (2+2p)+(1+p)=3+3 \\ &\iff 3+3p=6 \iff p=1 \end{aligned}$$

- Substituting into demand formula, equilibrium is

$$\begin{aligned} p &= (p_1,p_2)=(1,1), \\ x_1 &= (x_{11},x_{12})=(4,2), \\ x_2 &= (x_{21},x_{22})=(2,4) \end{aligned}$$

Example: interest rate

- One agent (or many identical agents), two periods ($t = 1, 2$), one physical good (apple)
- Recall we distinguish goods by time of availability, so mathematically it's just a static two good model
- Suppose endowments are $e = (e_1, e_2)$
- Suppose utility function is Cobb-Douglas:

$$u(x_1, x_2) = \log x_1 + \beta \log x_2$$

- Let $p = (p_1, p_2)$ be price vector
- Interpretation: p_1 is *spot price* (price to get one apple today); p_2 is *future price* (price to buy right to get one apple delivered tomorrow)

First, compute demand

- Initial wealth is $w = p_1 e_1 + p_2 e_2$
- Use Cobb-Douglas formula: caveat is coefficients must sum to 1
- Hence divide utility function by $(1+\beta)$:

$$v(x_1, x_2) = \frac{1}{1+\beta} \log x_1 + \frac{\beta}{1+\beta} \log x_2$$

(Recall monotonic transformation doesn't change behavior)

- Hence demand is

$$(x_1, x_2) = \left(\frac{1}{1+\beta} \frac{p_1 e_1 + p_2 e_2}{p_1}, \frac{\beta}{1+\beta} \frac{p_1 e_1 + p_2 e_2}{p_2} \right)$$

Market clearing condition

- Demand:

$$(x_1, x_2) = \left(\frac{1}{1+\beta} \frac{p_1 e_1 + p_2 e_2}{p_1}, \frac{\beta}{1+\beta} \frac{p_1 e_1 + p_2 e_2}{p_2} \right)$$

- Hence market clearing condition for good 1 is

$$\begin{aligned} x_1 = e_1 &\iff \frac{1}{1+\beta} \frac{p_1 e_1 + p_2 e_2}{p_1} = e_1 \\ &\iff \frac{p_2}{p_1} = \frac{\beta e_1}{e_2} \end{aligned}$$

- Get same conclusion if we use market clearing condition for good 2

Real interest rate

- Real interest rate is "how many apples I can get tomorrow by giving up one apple today"
- Let's do the calculation:
  - I sell one apple today: I receive $p_1 \times 1 = p_1$ of money
  - Using this amount, I can buy
  $$p_2 x = p_1 \iff x = \frac{p_1}{p_2}$$
  units of future contracts
  - With these future contracts, I get delivered p_1/p_2 apples tomorrow
- Hence real (gross) interest rate is
$$R = \frac{p_1}{p_2} = \frac{e_2}{\beta e_1}$$

Comparative statics of interest rate

- Real interest rate:
$$R = \frac{p_1}{p_2} = \frac{e_2}{\beta e_1}$$
- Hence $R \uparrow$ if
  - $\beta \downarrow$ (people impatient) or
  - $e_2/e_1 \uparrow$ (high economic growth)
- In many countries, interest rate has been declining for many decades
- Possible explanations:
  - population aging (people more patient)
  - lower economic growth

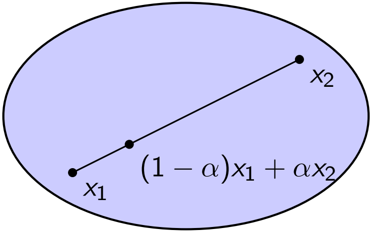
Chapter II

Convex Analysis and Convex Programming

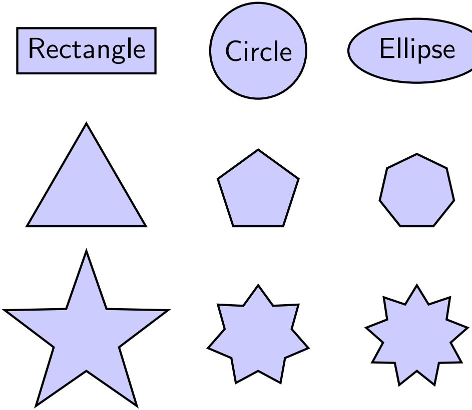
Convex set

- A set $C \subset \mathbb{R}^N$ is convex if any line segment joining any two points is entirely contained in C
- Mathematically, $C \subset \mathbb{R}^N$ is convex if

$$(1-\alpha)x_1 + \alpha x_2 \in C \text{ whenever } x_1, x_2 \in C \text{ and } 0 \le \alpha \le 1$$



Examples of convex and non-convex sets



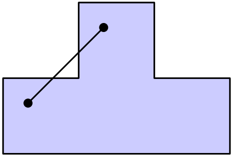
Convex

Convex

Non-convex

My favorite mathematical joke

- Chinese character for "convex" is not convex



Budget set is convex

- Let's prove that the budget set is convex
- Given price vector p and wealth w, budget set is

$$B = B(p, w) := \left\{ x \in \mathbb{R}^L_+ \,\middle|\, p \cdot x \le w \right\}$$

- Here is the proof: suppose $x_1, x_2 \in B$ and $0 \le \alpha \le 1$
- Then $x_1, x_2 \ge 0$, so $(1 - \alpha)x_1 + \alpha x_2 \ge 0$
- Also $p \cdot x_1 \le w$ and $p \cdot x_2 \le w$, so

$$\begin{aligned} p \cdot ((1 - \alpha)x_1 + \alpha x_2) &= (1 - \alpha)p \cdot x_1 + \alpha p \cdot x_2 \\ &\le (1 - \alpha)w + \alpha w = w \end{aligned}$$

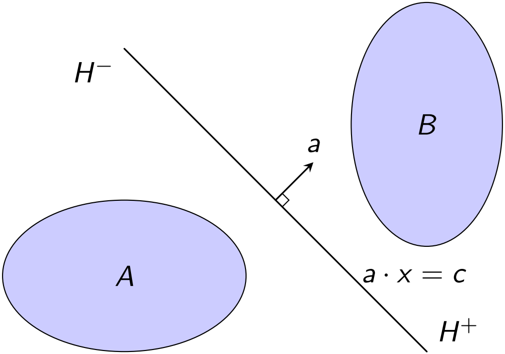
- Hence by definition $(1 - \alpha)x_1 + \alpha x_2 \in B$, so B is convex

Hyperplane

- Recall that equation of straight line in $\mathbb{R}^2$ has form
$$a_1x_1 + a_2x_2 = c$$
- Similarly, equation of plane in $\mathbb{R}^3$ has form
$$a_1x_1 + a_2x_2 + a_3x_3 = c$$
- More generally, hyperplane in $\mathbb{R}^N$ has form
$$a \cdot x = a_1x_1 + \dots + a_Nx_N = c$$
- If $=$ replaced with inequalities, we call halfspaces:
$$\begin{aligned} H^+ &= \left\{x \in \mathbb{R}^N \middle| a \cdot x \geq c\right\}, \\ H^- &= \left\{x \in \mathbb{R}^N \middle| a \cdot x \leq c\right\} \end{aligned}$$

Separation of convex sets

- We say two sets A, B can be separated if there is hyperplane (separating hyperplane) that puts A, B in opposite halfspaces
- $x \in A \implies a \cdot x \leq c$, $x \in B \implies a \cdot x \geq c$

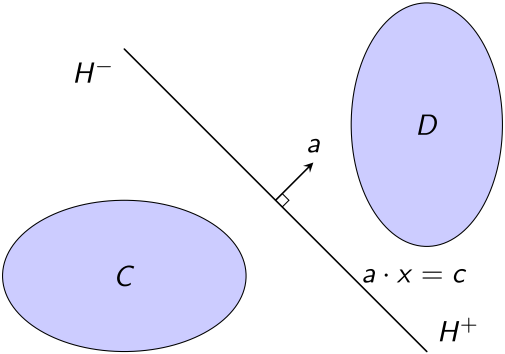


Separating Hyperplane Theorem (weak version)

Theorem

If $C, D \subset \mathbb{R}^N$ are nonempty, convex, and disjoint ($C \cap D = \emptyset$), then C, D can be separated: there exists $0 \neq a \in \mathbb{R}^N$ such that

$$\sup_{x \in C} a \cdot x \leq \inf_{y \in D} a \cdot y$$

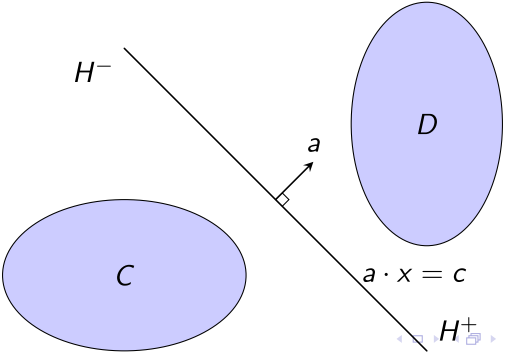


Separating Hyperplane Theorem (strong version)

Theorem

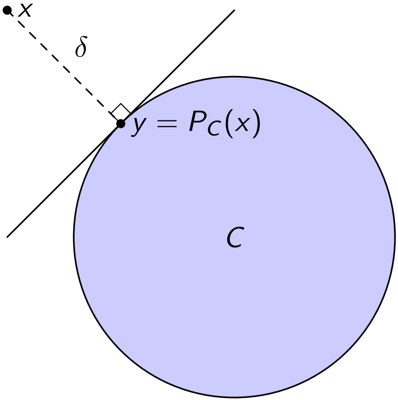
If $C, D \subset \mathbb{R}^N$ are nonempty, closed, convex, disjoint ($C \cap D = \emptyset$), and one of them is compact, then C, D can be strictly separated: there exists $0 \neq a \in \mathbb{R}^N$ such that

$$\sup_{x \in C} a \cdot x < \inf_{y \in D} a \cdot y$$



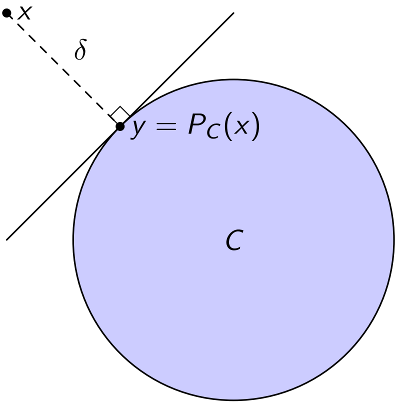
Proof of separating hyperplane theorem

- Rigorous proof is in lecture note
- Idea:
  1. If C is convex and x is any point, then there exists unique closest point (projection) $y = P_C(x)$



Proof of separating hyperplane theorem

- Idea:
  1. If C is convex and x is any point, then there exists unique closest point (projection) $y = P_C(x)$
  2. If x in exterior of C, then tangent hyperplane separates x and C



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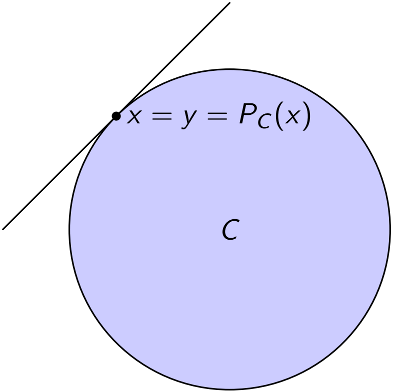
- Idea:
  1. If C is convex and x is any point, then there exists unique closest point (projection) $y = P_C(x)$
  2. If x in exterior of C, then tangent hyperplane separates x and C
  3. If x on boundary of C, same holds by limiting argument



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- Idea:
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  2. If x in exterior of C, then tangent hyperplane separates x and C
  3. If x on boundary of C, same holds by limiting argument
  4. If C, D are disjoint convex sets, define

     $$E := \{x - y \mid x \in C, y \in D\}$$

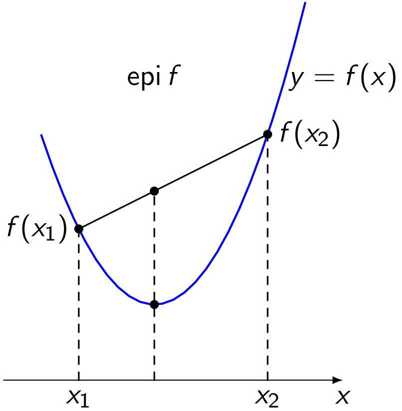
     Then E convex, $0 \notin E$, so can apply above argument to 0 and E (instead of x and C) to get conclusion

Convex functions

- For function $f : \mathbb{R}^N \to \mathbb{R}$, set of points on or above graph is called epigraph

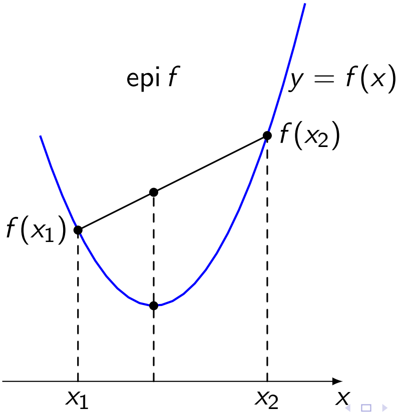
$$\operatorname{epi} f := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R} \,\middle|\, f(x) \leq y \right\}$$

- f is called convex function if $\operatorname{epi} f$ is convex set



Convex functions

- Alternative definition of convex function:
$$f((1-\alpha)x_1+\alpha x_2)\le(1-\alpha)f(x_1)+\alpha f(x_2)$$
for all $x_1, x_2 \in \mathbb{R}^N$ and $0 \le \alpha \le 1$
- If $N = 1$ and f twice differentiable, can show f convex if and only if $f'' \ge 0$



Concave functions

- We say $f : \mathbb{R}^N \to \mathbb{R}$ is concave function if $-f$ convex
- Equivalently, f concave if

$$f((1-\alpha)x_1 + \alpha x_2) \geq (1-\alpha)f(x_1) + \alpha f(x_2)$$

for all $x_1, x_2 \in \mathbb{R}^N$ and $0 \leq \alpha \leq 1$

Quasi-convex functions

- For function $f : \mathbb{R}^N \to \mathbb{R}$, set of points that give lower function value than given level is called lower contour set

$$L_f(y) := \left\{ x \in \mathbb{R}^N \,\middle|\, f(x) \le y \right\}$$

- f is called quasi-convex function if $L_f(y)$ is convex set for all y
- Equivalently, f quasi-convex if

$$f((1-\alpha)x_1 + \alpha x_2) \le \max\{f(x_1), f(x_2)\}$$

for all $x_1, x_2 \in \mathbb{R}^N$ and $0 \le \alpha \le 1$

Relation between convexity and quasi-convexity

- If f convex, then also quasi-convex
- Proof:
  - Take any y and suppose $x_1, x_2 \in L_f(y)$
  - By definition, $f(x_1) \le y$ and $f(x_2) \le y$
  - By convexity,

$$\begin{aligned} f((1-\alpha)x_1 + \alpha x_2) &\le (1-\alpha)f(x_1) + \alpha f(x_2) \\ &\le (1-\alpha)y + \alpha y = y \end{aligned}$$

  - Hence $(1-\alpha)x_1 + \alpha x_2 \in L_f(y)$, so f quasi-convex
- Not all quasi-convex functions are convex
  - Example: define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{|x|}$, then f quasi-convex but not convex (draw a picture)

Quasi-concave functions

- We say $f : \mathbb{R}^N \to \mathbb{R}$ is quasi-concave function if $-f$ quasi-convex
- Equivalently, f quasi-concave if upper contour set

$$U_f(y) := \left\{ x \in \mathbb{R}^N \,\middle|\, f(x) \geq y \right\}$$

is always convex

Convexity/concavity in economics

- Reasonable to assume utility functions are quasi-concave
- Because quasi-concavity of utility function implies convexity of upper contour set
- If an agent prefers x_1, x_2 to y, reasonable to assume he prefers mixture $(1-\alpha)x_1 + \alpha x_2$ to y
- Quasi-concavity nice because preserved by monotonic transformation (ordinal property)
  - If $u : \mathbb{R}^L_+ \to \mathbb{R}$ quasi-concave and $f : \mathbb{R} \to \mathbb{R}$ strictly increasing, then clearly $f \circ u$ quasi-concave
  - Concavity *not* preserved by monotonic transformation (cardinal property); example: $f(x) = \sqrt{x}$ (defined for $x \geq 0$) concave, but $f(x)^4 = x^2$ not concave

Constrained optimization problems

- In economics, we often encounter constrained optimization problems of the form

$$\begin{aligned} &\text{maximize} && f(x) \\ &\text{subject to} && g_k(x) \ge 0, && (k = 1, \dots, K) \end{aligned}$$

- Here f: objective function, $g_k(x) \ge 0$: constraint
- Example: utility maximization problem

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && x \in B_i(p) \end{aligned}$$

- Recalling $x \in B_i(p)$ if $x_l \ge 0$ for all l and $p \cdot (x - e_i) \le 0$, constraint functions are

$$g_k(x) = \begin{cases} x_k & (k = 1, \dots, L) \\ p \cdot (e_i - x) & (k = K := L + 1) \end{cases}$$

Gradient, unconstrained optimization

- Recall that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and achieves maximum (or minimum) at $\bar{x}$, then $f'(\bar{x}) = 0$
- Similar for multi-dimensional case: if $f : \mathbb{R}^N \to \mathbb{R}$ is differentiable and achieves maximum (or minimum) at $\bar{x}$, then

$$\frac{\partial f}{\partial x_n}(\bar{x}) = 0 \text{ for all } n$$

- More compactly, $\nabla f(\bar{x}) = 0$, where

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{bmatrix}$$

is gradient of f

Karush-Kuhn-Tucker (KKT) theorem

Theorem (Karush-Kuhn-Tucker for quasi-concave functions)

Let f, g_k's be quasi-concave and differentiable.

1. *If $\bar{x}$ is a solution to the optimization problem*

$$\begin{aligned} &\text{maximize} && f(x) \\ &\text{subject to} && g_k(x) \geq 0, && (k = 1, \dots, K) \end{aligned}$$

   and there exists a point x_0 such that $g_k(x_0) > 0$ for all k, then there exists $\lambda = (\lambda_1, \dots, \lambda_K) \in \mathbb{R}^K_+$ such that

$$\begin{aligned} &(FOC) && \nabla f(\bar{x}) + \sum_{k=1}^{K} \lambda_k \nabla g_k(\bar{x}) = 0, \\ &(CS) && (\forall k)\ \lambda_k \geq 0,\ g_k(\bar{x}) \geq 0,\ \lambda_k g_k(\bar{x}) = 0 \end{aligned}$$

2. *Conversely, if $\bar{x}$ and λ satisfy first-order and complementary slackness conditions, and $\nabla f(\bar{x}) \neq 0$, then $\bar{x}$ is a solution*

How to use KKT

1. Rewrite problem as

$$\begin{aligned} &\text{maximize} && f(x) \\ &\text{subject to} && g_k(x) \geq 0, \qquad (k = 1, \dots, K) \end{aligned}$$

   Constraints should be of form $g(x) \geq 0$!
2. Define Lagrangian

$$L(x, \lambda) = f(x) + \sum_{k=1}^{K} \lambda_k g_k(x)$$

3. Pretend you are maximizing L over x, and take first-order condition

$$0 = \nabla_x L = \nabla f + \sum_{k=1}^{K} \lambda_k \nabla g_k$$

4. Complementary slackness is just $\lambda_k g_k = 0$
5. Solve for x, λ that satisfy all conditions

Cobb-Douglas demand

- Consider Cobb-Douglas utility function

$$u(x) = \sum_{l=1}^{L} \alpha_l \log x_l,$$

where $\alpha_l > 0$ and $\sum_{l=1}^{L} \alpha_l = 1$

- Let's maximize subject to budget constraint $p \cdot x \leq w$
- Problem is equivalent to

$$\begin{aligned} &\text{maximize} && f(x) := \sum_{l=1}^{L} \alpha_l \log x_l \\ &\text{subject to} && g(x) := w - \sum_{l=1}^{L} p_l x_l \geq 0 \end{aligned}$$

(Can ignore nonnegativity constraints $x_l \geq 0$ because $\log 0 = -\infty$)

Cobb-Douglas demand

- Lagrangian is
$$L(x,\lambda)=\sum_{l=1}^{L}\alpha_l\log x_l+\lambda\left(w-\sum_{l=1}^{L}p_lx_l\right)$$
- First-order condition:
$$0=\frac{\partial L}{\partial x_l}=\frac{\alpha_l}{x_l}-\lambda p_l \iff x_l=\frac{\alpha_l}{\lambda p_l}$$
- It can't be $\lambda=0$, for otherwise $x_l=\infty$
- Hence $\lambda>0$, and complementary slackness implies
$$w=p\cdot x=\sum_{l=1}^{L}\frac{\alpha_l}{\lambda}=\frac{1}{\lambda}\iff\lambda=\frac{1}{w}$$
- Hence demand is $x_l=\frac{\alpha_l w}{p_l}$
- Can compute demand for CES in a similar way

Chapter III

Walras law

Strong monotonicity

- Suppose there are L goods, consumption bundle denoted by $x = (x_1, \dots, x_L)$
- Let $u : \mathbb{R}_+^L \to \mathbb{R}$ be a utility function
- Usually we like more to less, so $u(x) = u(x_1, \dots, x_L)$ strictly increasing in each x_l
- More compactly, we say u is strongly monotonic if $x < y$ implies $u(x) < u(y)$ (recall vector inequality from Ch. 1)

Weak monotonicity

- Strong monotonicity is a strong assumption because agents like to consume more no matter how much they are consuming
- Unrealistic, because after eating 100 hamburgers, no one wants to eat another hamburger
- We say u is weakly monotonic if $x \le y$ implies $u(x) \le u(y)$ and $x \ll y$ implies $u(x) < u(y)$
- So consuming more doesn't hurt, and consuming more of *all* goods is better

Local nonsation

- Weak monotonicity still a strong assumption because "bads" are ruled out
- No one wants to accept garbage, junk car, and nuclear waste for free
- We say u is locally nonsatiated if for any $x \in \mathbb{R}^L_+$ and $\epsilon > 0$, there exists y with $\|y - x\| < \epsilon$ and $u(y) > u(x)$
- Intuitively, local nonsatiation means that no matter what you are consuming, there is an arbitrarily close consumption bundle that you strictly prefer

Relation between three concepts

- Easy to show

$$\begin{aligned}\text{Strong monotonicity} &\implies \text{Weak monotonicity}\\ &\implies \text{Local nonsatiation}\end{aligned}$$

- For theoretical purpose, we will assume only local nonsatiation
- Examples:
  - Cobb-Douglas and CES utilities are strongly monotonic
  - Leontief utility is weakly monotonic but not strongly monotonic

Locally nonsatiated agent spends all income

Proposition

Let $u : \mathbb{R}^L_+ \to \mathbb{R}$ *be a locally nonsatiated (LNS) utility function and* $x(p, w)$ *be a solution to the utility maximization problem (UMP)*

$$\begin{aligned} &\text{maximize} && u(x) \\ &\text{subject to} && p \cdot x \leq w, \ x \in \mathbb{R}^L_+. \end{aligned}$$

Then $p \cdot x(p, w) = w$*.*

If u: LNS, then solution to UMP satisfies $p \cdot x = w$

Proof.

- Suppose u: LNS and $x = x(p, w)$ be a solution to UMP
- Suppose to the contrary that $p \cdot x < w$
- Take $\epsilon > 0$ such that $p \cdot y < w$ whenever $\|y - x\| < \epsilon$; possible because $x \mapsto p \cdot x$ continuous
- By LNS, can choose such y such that $u(y) > u(x)$
- So y is affordable and gives higher utility than x, contradiction

$\square$

Walras law

- Immediate corollary of above proposition is Walras law
- Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be Arrow-Debreu economy
  - $e_i \in \mathbb{R}^L_+$: endowment of agent i
  - $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ utility of agent i
- Let $p \in \mathbb{R}^L_+$ be any price vector and $p \cdot e_i$ be income of agent i
- Assume demand $x_i(p, p \cdot e_i)$ exists
- Define aggregate excess demand by

$$z(p) := \sum_{i=1}^{I} (x_i(p, p \cdot e_i) - e_i)$$

Walras law

Corollary

If each u_i locally nonsatiated, then

$$p \cdot z(p) = 0$$

Walras law

Corollary

If each u_i locally nonsatiated, then

$$p \cdot z(p) = 0$$

Proof.

- Let $x_i = x_i(p, p \cdot e_i)$
- By previous proposition, $p \cdot x_i = p \cdot e_i$, so $p \cdot (x_i - e_i) = 0$
- Hence

$$\begin{aligned} p \cdot z(p) &= p \cdot \sum_{i=1}^{I} (x_i - e_i) \\ &= \sum_{i=1}^{I} p \cdot (x_i - e_i) = 0 \end{aligned}$$

Implication of Walras law

- Under local nonsatiation (which is quite weak), Walras law implies

$$0 = p \cdot z(p) = \sum_{l=1}^{L} p_l z_l(p),$$

  where $z_l(p) = \sum_{i=1}^{I}(x_{il} - e_{il})$ aggregate excess demand of good l
- Hence if $p_l > 0$ for all l and $z_l(p) = 0$ (demand = supply) for all but one l, then market clears for the remaining one market
- This justifies checking only $L - 1$ market clearing conditions instead of L
- We now make this more precise

In equilibrium, excess supply means free good

Proposition

Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with LNS utilities and $\{p, (x_i)\}$ be a Walrasian equilibrium, where $p \geq 0$. Then

$$p_l \sum_{i=1}^{I} (x_{il} - e_{il}) = 0$$

for all l. In particular, $p_l = 0$ if $\sum_{i=1}^{I} (x_{il} - e_{il}) < 0$.

- Goods in excess supply are free
- Example: air

Proof

- By market clearing, $\sum_{i=1}^{I}(x_{il} - e_{il}) \le 0$ for all l
- Multiplying both sides by $p_l \ge 0$, get $p_l \sum_{i=1}^{I}(x_{il} - e_{il}) \le 0$ for all l
- Summing over l, get
$$\sum_{l=1}^{L} p_l \sum_{i=1}^{I}(x_{il} - e_{il}) \le 0$$
- But
$$0 = p \cdot z(p) = \sum_{l=1}^{L} p_l \sum_{i=1}^{I}(x_{il} - e_{il})$$
by Walras law, so all inequalities must hold with equality
- Hence $p_l \sum_{i=1}^{I}(x_{il} - e_{il}) = 0$ for all l

With strong monotonicity, can ignore one market

Theorem

Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with LNS utilities. Assume that at least one agent has a strongly monotonic preference. Let x_i be the demand of agent i given price p. Then $\{p, (x_i)\}$ is an equilibrium if and only if $p \gg 0$ and

$$\sum_{i=1}^{I} (x_{il} - e_{il}) = 0$$

for $l = 1, \dots, L - 1$.

- With strong monotonicity (for at least one agent), equilibrium implies demand = supply
- Furthermore, can ignore one market

Proof

- If $p_l = 0$, demand of strongly monotonic agent ∞, which is impossible
- So $p_l > 0$ for all l ($p \gg 0$) necessary
- In equilibrium, by previous proposition we know $p_l \sum_{i=1}^{I}(x_{il} - e_{il}) = 0$ for all l
- Since $p_l > 0$, it must be $\sum_{i=1}^{I}(x_{il} - e_{il}) = 0$ for all l
- Conversely, suppose $p \gg 0$ and $\sum_{i=1}^{I}(x_{il} - e_{il}) = 0$ for $l = 1, \dots, L-1$
- Then Walras law $p \cdot z(p) = 0$ implies $\sum_{i=1}^{I}(x_{il} - e_{il}) = 0$ for $l = L$, so equilibrium

Solving for equilibrium

- Above theorem can be used to compute equilibrium
- Here is algorithm:
  1. Check that agents have LNS utilities and at least one agent strongly monotonic
  2. For each $p \gg 0$, solve UMP

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \le w. \end{aligned}$$

  3. Normalize one price, say $p_1 = 1$; solve system of equation

$$\sum_{i=1}^{I} (x_{il}(p, p \cdot e_i) - e_{il}) = 0$$

     for p, where $l = 1, \dots, L - 1$
  4. Price vector is p, and demand is $x_i = x_i(p, p \cdot e_i)$

Edgeworth box economy with Cobb-Douglas utility

- Two agents, $I = \{1, 2\}$
- Two goods, $L = 2$
- Agent 1 has endowment $a = (a_1, a_2)$, utility

$$u_1(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2$$

- Agent 2 has endowment $b = (b_1, b_2)$, utility

$$u_2(x_1, x_2) = \beta \log x_1 + (1 - \beta) \log x_2$$

- What is equilibrium?

Edgeworth box economy with Cobb-Douglas utility

- Cobb-Douglas utility is strongly monotonic
- Normalize price vector to $(p_1, p_2) = (1, p)$
- Using Cobb-Douglas formula, demand of agent 1 is

$$(x_{11}, x_{12}) = \left(\alpha(a_1 + pa_2), \frac{(1-\alpha)(a_1 + pa_2)}{p}\right)$$

- Similarly, demand of agent 2 is

$$(x_{21}, x_{22}) = \left(\beta(b_1 + pb_2), \frac{(1-\beta)(b_1 + pb_2)}{p}\right)$$

- Market clearing condition for good 1 is

$$x_{11} + x_{21} = a_1 + b_1$$

Edgeworth box economy with Cobb-Douglas utility

- Market clearing condition for good 1 is

$$a_1 + b_1 = x_{11} + x_{21} = \alpha(a_1 + pa_2) + \beta(b_1 + pb_2)$$

- Solving for p, get

$$p = \frac{(1-\alpha)a_1 + (1-\beta)b_1}{\alpha a_2 + \beta b_2}$$

- Substitute this price into demand formula to compute demand

General Cobb-Douglas economy

- More generally, assume I agents, L goods, Cobb-Douglas utility
- Agent i has endowment e_i, utility

$$u_i(x) = \sum_{l=1}^{L} \alpha_{il} \log x_l,$$

  where $\sum_{l=1}^{L} \alpha_{il} = 1$
- Using Cobb-Douglas formula, demand of agent i is

$$x_{il} = \frac{\alpha_{il} p \cdot e_i}{p_l}$$

- Market clearing condition for good l is

$$e_l := \sum_{i=1}^{I} e_{il} = \sum_{i=1}^{I} x_{il} = \sum_{i=1}^{I} \frac{\alpha_{il} p \cdot e_i}{p_l}$$

General Cobb-Douglas economy

- Market clearing condition equivalent to

$$p \cdot \sum_{i=1}^{I} \alpha_{il} e_i = p_l e_l$$

- Normalizing one price (say $p_L = 1$), just a system of linear equation
- Can show unique $p \gg 0$ exists (after normalization), though proof not so easy (try if you can)

Chapter IV

Quasi-linear model

Solving for equilibrium

- We have seen that solving for equilibrium is generally complicated:
    1. for each agent i, solve utility maximization problem to express demand $x_i(p)$ as a function of price p
    2. solve for market clearing condition

$$\sum_{i=1}^{I} x_i(p) = \sum_{i=1}^{I} e_i$$

- In a "quasi-linear" model, it turns out that solving for equilibrium is relatively straightforward
- Furthermore, equilibrium has certain welfare property

Quasi-linear utility

- We say that a utility function u defined on $\mathbb{R} \times \mathbb{R}_+^L$ is quasi-linear if u has the form
$$u(x_0, x_1, \dots, x_L) = x_0 + \phi(x_1, \dots, x_L)$$
for some function $\phi : \mathbb{R}_+^L \to \mathbb{R}$
- Here there is a special good 0, called numéraire ("unit of account" in French), that can be consumed in positive or negative amounts
- Utility is additively separable between good 0 and the rest, and the good 0 part is linear (hence the name quasi-linear)
- Can think of good 0 as money or gold

Demand with quasi-linear utility

- Computing demand for quasi-linear utility is relatively straightforward
- Consider

$$\begin{aligned} &\text{maximize} && x_0 + \phi(x) \\ &\text{subject to} && x_0 + p \cdot x \leq w, \end{aligned}$$

where

  - $x = (x_1, \dots, x_L)$: bundle of non-numéraire goods,
  - $p = (p_1, \dots, p_L)$: price vector of non-numéraire goods,
  - w: income,
  - $p_0 =$ price of numéraire $= 1$

Demand with quasi-linear utility

- Consider

$$\begin{aligned} &\text{maximize} && x_0 + \phi(x) \\ &\text{subject to} && x_0 + p \cdot x \le w \end{aligned}$$

- Clearly utility locally nonsatiated (because strictly increasing in x_0), so may assume $x_0 + p \cdot x = w$
- Eliminating x_0, suffices to solve

$$\max_x [\phi(x) + w - p \cdot x]$$

- w just additive constant, so suffices to solve

$$\max_x [\phi(x) - p \cdot x]$$

Demand with quasi-linear utility

Proposition

Solution to

$$\begin{aligned} &\text{maximize} && x_0 + \phi(x) \\ &\text{subject to} && x_0 + p \cdot x \le w \end{aligned}$$

is $(x_0, x) = (w - p \cdot x(p), x(p))$*, where* $x(p) \in \mathbb{R}^L_+$ *solves*

$$\max_x [\phi(x) - p \cdot x]$$

- Demand of non-numéraire goods depend only on price vector p, independent of income w

Example: nonlinear part additively separable

- Suppose ϕ additively separable, so $\phi(x) = \phi_1(x_1) + \cdots + \phi_L(x_L)$
- Then
$$\phi(x) - p \cdot x = \sum_{l=1}^{L} (\phi_l(x_l) - p_l x_l)$$
- Hence maximizing $\phi(x) - p \cdot x$ equivalent to maximizing $\phi_l(x_l) - p_l x_l$ for each l
- First-order condition: $\phi_l'(x_l) - p_l = 0$
- Hence demand $x_l(p) = (\phi_l')^{-1}(p_l)$ depends only on own price
- Formal justification of "demand curve" in intermediate micro

Some caution

- Quasi-linear utility popular in microeconomics because it allows focusing on one market (partial equilibrium analysis)
- However, quasi-linear utility unrealistic because demand independent on income
- In reality, demand depends on income: as we get richer,
  - dormitory $\to$ apartment $\to$ house,
  - crappy used vehicle $\to$ ordinary new vehicle $\to$ Tesla, etc.

Quasi-linear economy

- $\mathcal{E} = \{I, (e_{i0}, e_i), u_i\}$ is quasi-linear economy if
  - $I = \{1, \dots, I\}$ is set of agents,
  - e_{i0} is i's endowment of numéraire good,
  - $e_i = (e_{i1}, \dots, e_{iL})$ is i's endowment vector of non-numéraire goods,
  - $u_i(x_0, x) = x_0 + \phi_i(x)$ is i's quasi-linear utility function, where $x = (x_1, \dots, x_L)$
- Definition of equilibrium standard
  1. Agent optimization
  2. Market clearing
- Let $\{(1, p), (x_{i0}, x_i)\}$ be equilibrium, where
  - $p = (p_1, \dots, p_L)$ price vector of non-numéraire goods,
  - x_{i0}: i's demand of numéraire good,
  - $x_i \in \mathbb{R}^L_+$: i's demand of non-numéraire goods

Equilibrium maximizes sum of utility

Theorem

$\mathcal{E} = \{I, (e_{i0}, e_i), u_i\}$ be quasi-linear economy with $u_i(x_0, x) = x_0 + \phi_i(x)$, where ϕ_i: continuous, differentiable, concave, and $\partial \phi_i / \partial x_l \to \infty$ as $x_l \to 0$. If $\{(1, p), (x_{i0}, x_i)\}$ equilibrium, then (x_i) solves

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^{I} \phi_i(y_i) \\ &\text{subject to} && \sum_{i=1}^{I} (y_{il} - e_{il}) \le 0 \quad (l = 1, \dots, L), \end{aligned}$$

where p_l is Lagrange multiplier.

- Equilibrium allocation of non-numéraire goods maximizes sum of utility subject to feasibility constraint

Proof

- In equilibrium, agents maximize utility
- Hence by previous proposition, $x_i \in \mathbb{R}^L_+$ solves

$$\max_x [\phi_i(x) - p \cdot x]$$

- First-order condition is $\nabla \phi_i(x_i) - p = 0$
- By market clearing, we know

$$p_l \sum_{i=1}^{I} (x_{il} - e_{il}) = 0$$

Proof

- Lagrangian of "maximize sum of utility" is

$$L(y_1, \dots, y_I, \lambda) = \sum_{i=1}^{I} \phi_i(y_i) + \lambda \cdot \sum_{i=1}^{I} (e_i - y_i),$$

  where $\lambda = (\lambda_1, \dots, \lambda_L)$
- FOC with respect to y_i is $\nabla \phi_i(y_i) - \lambda = 0$
- Hence recalling $\nabla \phi_i(x_i) - p = 0$, FOC satisfied by setting $y_i = x_i$ and $\lambda = p$
- $p_l \sum_{i=1}^{I} (x_{il} - e_{il}) = 0$ is precisely complementary slackness for "maximize sum of utility"
- By KKT theorem (Ch. 2), concavity implies sufficiency of FOC and CS

Maximizing sum of utility yields equilibrium

Theorem

$\mathcal{E} = \{I, (e_{i0}, e_i), u_i\}$ be quasi-linear economy with $u_i(x_0, x) = x_0 + \phi_i(x)$, where ϕ_i: continuous, differentiable, concave, and $\partial \phi_i / \partial x_l \to \infty$ as $x_l \to 0$. Suppose (x_i) solves

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^{I} \phi_i(y_i) \\ &\text{subject to} && \sum_{i=1}^{I} (y_{il} - e_{il}) \le 0 \quad (l = 1, \dots, L), \end{aligned}$$

where p_l is Lagrange multiplier. Then $\{(1, p), (x_{i0}, x_i)\}$ is equilibrium, where $p = (p_1, \dots, p_L)$ and $x_{i0} = e_{i0} + p \cdot (e_i - x_i)$.

- Allocation of non-numéraire goods that maximizes sum of utility subject to feasibility constraint is equilibrium

Proof

- Lagrangian of "maximize sum of utility" is

$$L(y_1, \dots, y_I, p) = \sum_{i=1}^{I} \phi_i(y_i) + p \cdot \sum_{i=1}^{I} (e_i - y_i),$$

  where $p = (p_1, \dots, p_L)$ is Lagrange multiplier
- FOC with respect to y_i (evaluated at x_i) is $\nabla \phi_i(x_i) - p = 0$
- Since ϕ_i concave, x_i solves

$$\max_x [\phi_i(x) - p \cdot x]$$

- Hence by previous proposition, x_i is demand for non-numéraire goods for price p
- Budget constraint implies $x_{i0} = e_{i0} + p \cdot (e_i - x_i)$ is demand for numéraire good

Proof

- Feasibility implies
$$\sum_{i=1}^{I}(x_{il}-e_{il}) \le 0$$
for all l, hence markets for non-numéraire goods clear
- Complementary slackness implies
$$p_l \sum_{i=1}^{I}(x_{il}-e_{il}) = 0$$
for all l
- Hence
$$\sum_{i=1}^{I} x_{i0} = \sum_{i=1}^{I}(e_{i0}+p\cdot(e_i-x_i)) = \sum_{i=1}^{I} e_{i0},$$
so market for numéraire good also clears

Summary of theoretical results

- Equilibrium allocation maximizes sum of utility
- Conversely, allocation that maximizes sum of utility is equilibrium
- Hence equilibrium is "desirable" in particular sense
- Mathematical formulation of Jeremy Bentham's "greatest happiness principle":

  > *Fundamental axiom … it is the greatest happiness of the greatest number that is the measure of right and wrong*

  https://en.wikipedia.org/wiki/Jeremy_Bentham
- Note: result specific to quasi-linear economy

Example

- Consider I-agent quasi-linear economy with one non-numéraire good, where utility function is

$$u_i(x_0, x_1) = x_0 + \beta_i \log x_1$$

- Nonlinear part is $\phi_i(x) = \beta_i \log x$
- Computation of equilibrium reduces to solving

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^I \beta_i \log x_i \\ &\text{subject to} && \sum_{i=1}^I (x_i - e_i) \le 0 \end{aligned}$$

Example

- Let $e = \sum_{i=1}^{I} e_i$ be aggregate endowment
- Lagrangian is

$$L = \sum_{i=1}^{I} \beta_i \log x_i + p \left(e - \sum_{i=1}^{I} x_i \right)$$

- First-order condition with respect to x_i:

$$\frac{\beta_i}{x_i} - p = 0 \iff x_i = \frac{\beta_i}{p}$$

- Complementary slackness:

$$\sum_{i=1}^{I} x_i = e \iff p = \frac{1}{e} \sum_{i=1}^{I} \beta_i$$

- Hence demand is $x_i = e \frac{\beta_i}{\sum_{i=1}^{I} \beta_i}$

Chapter V

Welfare properties of equilibrium

Is equilibrium "desirable"?

- In previous chapter, we learned that with quasi-linear utilities, equilibrium allocation maximizes sum of utilities
- Hence equilibrium is "desirable" in some sense
- This chapter proves two important results:
  1. First welfare theorem: equilibrium allocation is efficient
  2. Second welfare theorem: any efficient allocation can be achieved as equilibrium, after appropriate direct tax/subsidy
- First and second welfare theorems are strong defense of capitalism

How to define "desirability": unanimous rule

- Consider and economy with I agents and L goods: agent i has utility function $u_i : \mathbb{R}_+^L \to \mathbb{R}$
- Let $x = (x_i)_{i=1}^I$ and $y = (y_i)_{i=1}^I$ be two allocations
- How can we say allocation y is better than x?
- Problem: people have different opinions
  - Some people (like me) may prefer little government intervention
  - Others may prefer a lot of government intervention
- Hence we give up ranking any two allocations x, y, and we rank only those that we agree unanimously:

"y is (socially) better than x if all agents prefer y to x"

Pareto dominance

Definition

Let $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ be agent i's utility function and $x = (x_i)_{i=1}^I, y = (y_i)_{i=1}^I$ be two allocations. Then

- $y = (y_i)_{i=1}^I$ weakly Pareto dominates $x = (x_i)_{i=1}^I$ if
$$u_i(y_i) \geq u_i(x_i) \text{ for all } i$$
- $y = (y_i)_{i=1}^I$ strictly Pareto dominates $x = (x_i)_{i=1}^I$ if
$$u_i(y_i) > u_i(x_i) \text{ for all } i$$
- $y = (y_i)_{i=1}^I$ Pareto dominates $x = (x_i)_{i=1}^I$ if
$$\begin{aligned} &u_i(y_i) \geq u_i(x_i) \text{ for all } i \text{ and} \\ &u_i(y_i) > u_i(x_i) \text{ for some } i \end{aligned}$$

(In some sense, similar to how vector inequalities $\geq, \gg, >$ are defined, though for vector of utilities, not consumption)

Pareto efficiency

- Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be Arrow-Debreu economy
  - $e_i \in \mathbb{R}^L_+$: endowment of agent i
  - $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ utility of agent i
- Let $x = (x_i)_{i=1}^I$ be a feasible allocation, so

$$\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i$$

- We want to ask whether x is desirable

Pareto efficiency

Definition

A feasible allocation $x = (x_i)_{i=1}^I$ is Pareto efficient (or just efficient) if it is not Pareto dominated by any other feasible allocation $y = (y_i)_{i=1}^I$

- A feasible allocation is inefficient if it is Pareto dominated by some other feasible allocation, otherwise efficient
- Another way to understand: a situation is inefficient if rejected by unanimous voting, otherwise efficient
- Yet another way to understand: a situation is inefficient if we can make somebody better off without hurting anybody, otherwise efficient
- If x inefficient, we say y is Pareto improvement if y is feasible and Pareto dominates x

Example

- Suppose there is only one good (cake), and we all like cake
  - Allocation A: I eat all the cake, you eat nothing
  - Allocation B: we all share the cake equally
  - Allocation C: I eat half the cake, we throw away the remaining half
- Then A, B are both efficient, because to make somebody better off (give more cake), we need to take cake from somebody else (which hurts that person)
- C is inefficient, because we can make Pareto improvement if we share remaining half
- Note: Pareto efficiency is a weak concept because it passes only unanimous voting
- It disregards many aspects (e.g., equity)

First welfare theorem

Theorem

Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be economy with locally nonsatiated utilities. If $\{p, (x_i)\}$ is an equilibrium, then $(x_i)_{i=1}^I$ is Pareto efficient.

- Extremely important
- It essentially says "capitalism is great"—market achieves efficient outcome without government intervention
- Mathematical formulation of Adam Smith's "invisible hand"

Quote from Adam Smith

As every individual, therefore, endeavours as much as he can both to employ his capital in the support of domestic industry, and so to direct that industry that its produce may be of the greatest value; every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it. By preferring the support of domestic to that of foreign industry, he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.

Modern and plain translation of Adam Smith

- People are selfish
- When people pursue their self interest, they promote social welfare as if led by an invisible hand
- But promoting social welfare was not intentional
- When government intends to promote social welfare, it messes up

Proof of first welfare theorem

- Though a super important theorem, proof not particularly difficult (you should be able to replicate; proficiency in logical argument necessary)
- As is often the case, we prove by contradiction
- So suppose $\{p, (x_i)\}$ is an equilibrium but (x_i) inefficient
- By definition, there exists feasible allocation (y_i) such that (y_i) Pareto dominates (x_i)
- By definition,
    - $\sum_{i=1}^{I} y_i \leq \sum_{i=1}^{I} e_i$,
    - $u_i(y_i) \geq u_i(x_i)$ for all i,
    - $u_i(y_i) > u_i(x_i)$ for some i

Proof of first welfare theorem

- Consider an agent with $u_i(y_i) \geq u_i(x_i)$ (every agent)
- Claim: $p \cdot y_i \geq p \cdot e_i$
- Here is the proof (again by contradiction)
  1. Suppose $p \cdot y_i < p \cdot e_i$
  2. By local nonsatiation, there exists y' with $u_i(y') > u_i(y_i) \geq u_i(x_i)$ and $p \cdot y' < p \cdot e_i$
  3. So y' affordable but gives higher utility than x_i
  4. Contradiction because x_i maximizes utility within budget
  5. Hence supposition $p \cdot y_i < p \cdot e_i$ is false, and hence $p \cdot y_i \geq p \cdot e_i$

Proof of first welfare theorem

- Consider an agent with $u_i(y_i) > u_i(x_i)$ (at least one agent)
- Claim: $p \cdot y_i > p \cdot e_i$
- Here is the proof (again by contradiction)
  1. Suppose $p \cdot y_i \leq p \cdot e_i$
  2. Then y_i affordable but gives higher utility than x_i
  3. Contradiction because x_i maximizes utility within budget
  4. Hence supposition $p \cdot y_i \leq p \cdot e_i$ is false, and hence $p \cdot y_i > p \cdot e_i$

Proof of first welfare theorem

- We know by feasibility that $\sum_{i=1}^{I} y_i \le \sum_{i=1}^{I} e_i$
- We have shown that
  - $p \cdot y_i \ge p \cdot e_i$ for all i,
  - $p \cdot y_i > p \cdot e_i$ for some i
- By summing over i, get

$$\begin{aligned}
p \cdot \sum_{i=1}^{I} y_i &= \sum_{i=1}^{I} p \cdot y_i && (\because \text{exchange sum and inner product}) \\
&> \sum_{i=1}^{I} p \cdot e_i && (\because p \cdot y_i \ge p \cdot e_i, \text{ with at least one } >) \\
&= p \cdot \sum_{i=1}^{I} e_i && (\because \text{exchange sum and inner product}) \\
&\ge p \cdot \sum_{i=1}^{I} y_i && (\because \text{feasibility and } p \ge 0),
\end{aligned}$$

contradiction $\square$

Proof of first welfare theorem

- Above (simple) proof of first welfare theorem is due to two independent papers by Arrow and Debreu in 1951
- But an earlier but more complicated proof (that assumes all sorts of unnecessary assumptions such concavity, differentiability, etc.) is due to Oskar Lange in 1942

Proof of first welfare theorem

- Above (simple) proof of first welfare theorem is due to two independent papers by Arrow and Debreu in 1951
- But an earlier but more complicated proof (that assumes all sorts of unnecessary assumptions such concavity, differentiability, etc.) is due to Oskar Lange in 1942
- Ironic, because first welfare theorem says “capitalism is great” but Lange was a communist

Local nonsatiation is necessary

- First welfare theorem assumes only local nonsatiation (and equilibrium)
- We cannot do away with local nonsatiation
- Counterexample:
  - Two agents, two goods, with $e_1 = e_2 = (5, 5)$ and

$$u_1(x_1, x_2) = x_1 x_2,$$
$$u_2(x_1, x_2) = \min\{x_1 x_2, 16\}$$

  - Then clearly $\{p, (x_i)\} = \{(1, 1), (5, 5), (5, 5)\}$ is equilibrium (why?)
  - But allocation $y_1 = (6, 6)$, $y_2 = (4, 4)$ Pareto dominates $x_1 = (5, 5)$, $x_2 = (5, 5)$

Key to the proof of first welfare theorem

- As we see from above, local nonsatiation is key to proof of first welfare theorem, but LNS is weak assumption
- One strong implicit assumption is that market is complete (all goods are traded)
- Complete market allows agents to use a single budget constraint of the form $p \cdot x \leq w$, which is key to proof
- Without complete market, proof breaks down
- Hence first welfare theorem need not hold in reality because not all goods traded (e.g., future income)

Equilibrium with transfer payments

- First welfare theorem states that market mechanism achieves an efficient allocation of resources
- We now aim to prove second welfare theorem, which is a partial converse—any efficient allocation can be achieved as equilibrium after appropriate taxes and transfers
- For this we define an equilibrium with transfer payments

Equilibrium with transfer payments

Definition

Let $\{I, (e_i), (u_i)\}$ be an Arrow-Debreu economy. A price p, an allocation (x_i), and transfer payments (t_i) constitute a competitive equilibrium with transfer payments if

1. (Agent optimization) for each i, x_i solves

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \leq p \cdot e_i - t_i, \end{aligned}$$

2. (Market clearing) $\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i$,
3. (Balanced budget) $\sum_{i=1}^{I} t_i = 0$.

- Essentially same as competitive equilibrium, except agent i pays tax t_i (receives subsidy $-t_i$ if $t_i < 0$)

Second welfare theorem

Theorem

Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with continuous, quasi-concave, locally nonsatiated utilities. If (x_i) is a feasible Pareto efficient allocation with $x_i \gg 0$ for all i, then there exist a price vector p and transfer payments (t_i) such that $\{p, (x_i), (t_i)\}$ is a competitive equilibrium with transfer payments.

- To achieve a specific Pareto efficient allocation, the government should not regulate markets but simply impose lump sum taxes, make lump sum transfers, and *laissez faire*
- Note differences in assumptions
  - First welfare theorem: u_i locally nonsatiated
  - Second welfare theorem: u_i continuous, quasi-concave, locally nonsatiated

Idea of proof of second welfare theorem

1. Unlike first welfare theorem (where equilibrium $\{p, (x_i)\}$ given), here only given allocation (x_i)
2. Hence need to construct price vector p: use separating hyperplane theorem
3. Once p constructed, define transfer t_i to satisfy budget constraint: $p \cdot x_i = p \cdot e_i - t_i$
4. Then show that $\{p, (x_i), (t_i)\}$ is equilibrium with transfer payment

Proof of second welfare theorem: construct p

- Define (strict) upper contour set of agent i by $U_i = \left\{ y \in \mathbb{R}^L_+ \,\middle|\, u_i(y) > u_i(x_i) \right\}$
  - Since u_i continuous, U_i is open
  - Since u_i quasi-concave, U_i is convex
  - Since u_i locally nonsatiated, U_i is nonempty
- Define set of aggregate demand in upper contour set by

$$U = \left\{ y = \sum_{i=1}^{I} y_i \,\middle|\, (\forall i) y_i \in U_i \right\}$$

- Since each U_i nonempty, open, and convex, so is U

Proof of second welfare theorem: construct p

- Define $E = \left\{ x \in \mathbb{R}^L \,\middle|\, x \le \sum_{i=1}^I e_i \right\}$, set of vectors bounded above by aggregate supply (endowment)
- Clearly E nonempty, convex
- Claim: $U \cap E = \emptyset$ (empty intersection)
- Here is the proof (again by contradiction)
  - If $y \in U \cap E$, by definition of U, E, we can take $y_i \in U_i$ such that $\sum_{i=1}^I y_i \le \sum_{i=1}^I e_i$
  - Then $u_i(y_i) > u_i(x_i)$, so agent i strictly prefers y_i
  - Hence allocation (y_i) (strictly) Pareto dominates (x_i)
  - But $\sum_{i=1}^I y_i \le \sum_{i=1}^I e_i$, so (y_i) feasible
  - Contradiction because (x_i) is Pareto efficient by assumption

Proof of second welfare theorem: construct p

- Recall
  - $U = \left\{ y = \sum_{i=1}^{I} y_i \,\middle|\, (\forall i) u_i(y_i) > u_i(x_i) \right\}$
  - $E = \left\{ x \in \mathbb{R}^L \,\middle|\, x \leq \sum_{i=1}^{I} e_i \right\}$
- We know U, E are nonempty, convex, and $U \cap E = \emptyset$
- Hence by separating hyperplane theorem, there exists nonzero vector $p \in \mathbb{R}^L$ such that

$$\sup_{x \in E} p \cdot x \leq \inf_{y \in U} p \cdot y$$

Proof of second welfare theorem: $p > 0$

- Claim: $p > 0$
- Here is the proof (again by contradiction)
  - Suppose $p_l < 0$ for some l
  - Recall $p \cdot x \le p \cdot y$ for all $x \in E$ and $y \in U$
  - Since $\sum_{i=1}^{I} e_i \geq 0$, can choose $x \in E$ with $x_l = -k < 0$ and $x_{l'} = 0$ for all $l' \neq l$
  - Then $p \cdot x = -p_l k \to \infty$ as $k \to \infty$
  - Since $p \cdot x > p \cdot y$ for large enough k, contradiction
  - Therefore $p_l \geq 0$ for all l, and since $p \neq 0$, it must be $p > 0$
- Now that $p > 0$, it serves as a price vector

Proof of second welfare theorem: construct t_i

- Define t_i to satisfy individual budget constraint, so $p \cdot x_i = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i)$
- Claim: $\sum_{i=1}^{I} t_i = 0$ (budget balance)
- Here is the proof
  - Since (x_i) feasible, we have $\sum_{i=1}^{I}(x_i - e_i) \leq 0$
  - Multiplying $p > 0$ as inner product, $p \cdot \sum_{i=1}^{I}(x_i - e_i) \leq 0$
  - Hence $\sum_{i=1}^{I} t_i \geq 0$
  - Setting $x = \sum_{i=1}^{I} e_i \in E$ in $p \cdot x \leq p \cdot y$, get $p \cdot \sum_{i=1}^{I} e_i \leq p \cdot \sum_{i=1}^{I} y_i$
  - By local nonsatiation, can choose $y_i \in U_i$ arbitrarily close to x_i
  - Hence $p \cdot \sum_{i=1}^{I} e_i \leq p \cdot \sum_{i=1}^{I} x_i$, implying $\sum_{i=1}^{I} t_i \leq 0$
  - Hence $\sum_{i=1}^{I} t_i = 0$

Proof of second welfare theorem: what remains

- (x_i) feasible, so market clearing $\sum_{i=1}^{I} x_i \le \sum_{i=1}^{I} e_i$ trivial
- Hence to show $\{p, (x_i), (t_i)\}$ is equilibrium with transfer payments, it remains to show x_i maximizes utility subject to budget constraint $p \cdot x \le p \cdot e_i - t_i$
- Hence suffices to show $p \cdot y_i > p \cdot e_i - t_i$ whenever $u_i(y_i) > u_i(x_i)$, for all i
- To prove this by contradiction, suppose $p \cdot y_i \le p \cdot e_i - t_i$ for some i and $y_i \in U_i$
- Without loss of generality, assume $i = 1$, so $p \cdot y_1 \le p \cdot e_1 - t_1 = p \cdot x_1$

Proof of second welfare theorem: agent optimization

- $p \cdot y_1 \leq p \cdot e_1 - t_1 = p \cdot x_1 > 0$ because $p > 0$ and $x_1 \gg 0$
- Since u_1 continuous and $u_1(y_1) > u_1(x_1)$, can choose z_1 with $u_1(z_1) > u_1(x_1)$ and $\epsilon = p \cdot x_1 - p \cdot z_1 > 0$
- For all $i \neq 1$, by local nonsatiation we can take z_i with $u_i(z_i) > u_i(x_i)$ and $p \cdot z_i < p \cdot x_i + \epsilon/I$
- Then $z_i \in U_i$ for all i, so using $p \cdot x \leq p \cdot y$ for $x = \sum_{i=1}^{I} e_i$ and $y = \sum_{i=1}^{I} z_i$, get

$$\begin{aligned}
\sum_{i=1}^{I} p \cdot e_i = p \cdot \sum_{i=1}^{I} e_i \leq p \cdot \sum_{i=1}^{I} z_i = \sum_{i=1}^{I} p \cdot z_i \\
\leq (p \cdot x_1 - \epsilon) + \sum_{i=2}^{I} (p \cdot x_i + \epsilon/I) = \sum_{i=1}^{I} p \cdot x_i - \frac{\epsilon}{I} \\
\implies \frac{\epsilon}{I} \leq \sum_{i=1}^{I} p \cdot (x_i - e_i) = -\sum_{i=1}^{I} t_i = 0, \text{ contradiction}
\end{aligned}$$

Limitation of second welfare theorem

- Second welfare theorem states that any Pareto efficient allocation can be achieved as an equilibrium outcome after appropriate direct taxes/transfers
- Hence a policy implication is that direct taxes (e.g., income tax) are preferable to indirect taxes (e.g., consumption tax)
- However, limitation of second welfare theorem is that tax amount t_i is individual specific: in reality, it is unrealistic to assume that government has information required to determine t_i
- Hence better to take second welfare theorem as theoretical, not practical (on the other hand, first welfare theorem has practical content)

Summary so far

- An allocation is Pareto efficient if it is impossible to make somebody better of without hurting somebody else
- First welfare theorem: competitive equilibrium allocation is Pareto efficient
  - Assumption: u_i locally nonsatiated
- Second welfare theorem: any Pareto efficient allocation can be achieved as competitive equilibrium after appropriate direct tax/transfer
  - Assumption: u_i continuous, quasi-concave, locally nonsatiated, and (x_i) interior

Characterizing Pareto efficient allocations

- How can we know an allocation (x_i) is Pareto efficient?
- We can use second welfare theorem
- Suppose for simplicity that each u_i differentiable and $\nabla u_i \gg 0$
- x_i solves

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \le w_i, \end{aligned}$$

  where income is $w_i = p \cdot e_i - t_i$
- Lagrangian is

$$L_i(x, \lambda) = u_i(x) + \lambda_i(w_i - p \cdot x)$$

- First order condition: $\nabla u_i(x_i) = \lambda_i p$

Characterizing Pareto efficient allocations

- Suppose (x_i) interior, Pareto efficient, and $\nabla u_i \gg 0$
- We know $\nabla u_i(x_i) = \lambda_i p$; since $\nabla u_i \gg 0$, it must be $\lambda_i > 0$ and $p \gg 0$
- Taking l-th entry, get $\partial u_i(x_i)/\partial x_l = \lambda_i p_l$
- Hence marginal rate of substitution (MRS)

$$\mathrm{MRS}_{i,lm}(x_i) = \frac{\partial u_i(x_i)/\partial x_l}{\partial u_i(x_i)/\partial x_m} = \frac{p_l}{p_m}$$

independent of individual i

Characterizing Pareto efficient allocations

Proposition

Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be an economy with quasi-concave utilities such that $\nabla u_i \gg 0$. Let (x_i) be an allocation such that $x_i \gg 0$ for all i. Then (x_i) is Pareto efficient if and only if $\sum_{i=1}^I x_i = \sum_{i=1}^I e_i$ and the marginal rate of substitution is equalized across agents.

Proof of (x_i) efficient $\implies$ MRS equalized

- Since u_i differentiable, in particular continuous
- Since $\nabla u_i \gg 0$, u_i strongly monotonic, so $\sum_{i=1}^I x_i = \sum_{i=1}^I e_i$ necessary for efficiency
- Since u_i continuous, quasi-concave, LNS, and $x_i \gg 0$, can apply second welfare theorem
- By argument using Lagrangian outlined above, MRS is equalized across agents

Proof of MRS equalized $\implies$ (x_i) efficient

- Conversely, suppose $x_i \gg 0$, $\sum_{i=1}^I x_i = \sum_{i=1}^I e_i$, and MRS equalized across agents
- Define $p_1 = 1$
- Define $p_l > 0$ by

$$p_l = \frac{p_l}{p_1} = \frac{\partial u_i(x_i)/\partial x_l}{\partial u_i(x_i)/\partial x_1} = \mathrm{MRS}_{i,l1}(x_i),$$

which is well-defined because MRS equalized across agents

Proof of MRS equalized $\implies (x_i)$ efficient

- Consider UMP with initial endowment x_i:

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \le p \cdot x_i \end{aligned}$$

- Lagrangian $L_i(x, \lambda_i) = u_i(x) + \lambda_i(p \cdot x_i - p \cdot x)$
- Define λ_i to satisfy FOC with respect to good 1:

$$\frac{\partial u_i(x_i)}{\partial x_1} = \lambda_i p_1 = \lambda_i$$

- Using definition of MRS and p_l, get

$$\frac{\partial u_i(x_i)}{\partial x_l} = p_l \frac{\partial u_i(x_i)}{\partial x_1} = \lambda_i p_l,$$

which is FOC with respect to good l

Proof of MRS equalized $\implies (x_i)$ efficient

- We know $\partial u_i(x_i)/\partial x_l = \lambda_i p_l$ for all l, hence x_i satisfies FOC for all goods
- Complementary slackness condition trivial because $p \cdot x_i - p \cdot x_i = 0$
- Since u_i quasi-concave and $\nabla u_i \gg 0$, by sufficiency result in KKT theorem, x_i solves UMP
- Claim: (x_i) Pareto efficient
- This is obvious from first welfare theorem, because
  - We know x_i solves utility maximization problem of agent i, given price vector p
  - $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i$, so markets clear
  - Hence $\{p, (x_i)\}$ is competitive equilibrium, so (x_i) Pareto efficient by first welfare theorem

Chapter VI

Computation of equilibrium

Computation of equilibrium

- Computing equilibrium is usually difficult (by hand) because
  - for each agent, need to solve utility maximization problem given price vector p, and
  - need to solve for price vector p that clears market, which is system of nonlinear equations
- We have seen several examples where computation of equilibrium is relatively straightforward
  - With quasi-linear utilities, suffices to maximize sum of utilities
  - With Cobb-Douglas utilities, market clearing condition is linear equation in p

Computation of equilibrium

- Although we can always solve for equilibrium numerically, examples of closed-form solutions are useful for building intuition and applications
- These examples include
  1. identical homothetic preferences with arbitrary endowments
  2. arbitrary homothetic preferences with collinear endowments
  3. hyperbolic absolute risk aversion (HARA) utility
- Here we focus on last case

Expected utility

- In anticipation of applications to finance, let us distinguish goods by state of the world (e.g., umbrella when sunny or rainy)
- One (physical) consumption good, but states labeled by $s = 1, 2, \dots, S$
- Probability of state s is $\pi_s > 0$
- Consider an agent with expected utility, so utility from consumption bundle $x = (x_1, \dots, x_S)$ is

$$U(x) = \mathrm{E}[u(x)] = \sum_{s=1}^{S} \pi_s u(x_s)$$

- Here $u : \mathbb{R}_+ \to \mathbb{R}$ is called the von Neumann-Morgenstern utility function

Risk aversion

- Suppose agent has vNM utility function u and initial wealth $w > 0$, where $u' > 0$ (increasing) and $u'' < 0$ (concave)
- Consider following two options
  1. the agent must take a small gamble, so wealth becomes $w + \epsilon$, where ϵ is a random variable with mean zero
  2. the agent can pay insurance premium a to avoid gamble
- When is agent indifferent between two options? Clearly

$$\mathrm{E}\, u(w + \epsilon) = u(w - a)$$

- Apply Taylor's theorem to approximate a

Risk aversion

- Recall Taylor's theorem

$$f(x+\Delta x) \approx \begin{cases} f(x) + f'(x)\Delta x & \text{(first-order)} \\ f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2 & \text{(second-order)} \end{cases}$$

- Letting $x = w$, $\Delta x = -a$, and using first-order approximation, get $u(w - a) \approx u(w) - u'(w)a$
- Letting $x = w$, $\Delta x = \epsilon$, and using second-order approximation, get

$$u(w + \epsilon) \approx u(w) + u'(w)\epsilon + \frac{1}{2} u''(w)\epsilon^2$$

- Taking expectations and using $\mathrm{E}[\epsilon] = 0$ and setting $\mathrm{E}[\epsilon^2] = \mathrm{Var}[\epsilon]$, get

$$\mathrm{E}[u(w + \epsilon)] \approx u(w) + \frac{1}{2} u''(w) \mathrm{Var}[\epsilon]$$

Risk aversion

- Hence insurance premium a is approximately

$$\begin{aligned} u(w)-u'(w)a &\approx u(w-a) \\ &= \mathrm{E}[u(w+\epsilon)] \approx u(w)+\frac{1}{2}u''(w)\operatorname{Var}[\epsilon] \\ \implies a &\approx \frac{1}{2}\operatorname{Var}[\epsilon]\left(-\frac{u''(w)}{u'(w)}\right) \end{aligned}$$

- Insurance premium is proportional to variance and quantity $-u''(w)/u'(w)$, which is called (absolute) risk aversion of u at w
- Reciprocal of (absolute) risk aversion, $-u'(w)/u''(w)$, is called (absolute) risk tolerance

Hyperbolic absolute risk aversion

- In many applications, it is often convenient to use utility functions with linear risk tolerance (LRT)

$$-\frac{u'(x)}{u''(x)} = ax + b$$

- LRT utility functions has hyperbolic absolute risk aversion (HARA)

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax+b},$$

  where x is in the range $ax + b > 0$

Characterization of HARA utility

- Straightforward to characterize HARA utility by integration
- Assume $a \neq 0$ and if we integrate

  $$\frac{u''(x)}{u'(x)} = -\frac{1}{ax+b},$$

  we get

  $$\log u'(x) = -\frac{1}{a}\log(ax+b) + \text{const.}$$

- Taking exponential, get $u'(x) = C(ax+b)^{-1/a}$ for some $C > 0$
- Assuming $a \neq 1$ and integrating once again, get

  $$u(x) = C\frac{1}{a-1}(ax+b)^{1-1/a} + D$$

  for some constants $C > 0$ and $D \in \mathbb{R}$

Remaining cases: $a = 0, 1$

- If $a = 0$, integrating $u''(x)/u'(x) = -1/b$, get $\log u'(x) = -x/b + \text{const.}$
- Taking exponential, get $u'(x) = C\mathrm{e}^{-x/b}$ for some $C > 0$
- Integrating once again, get
$$u(x) = -Cb\mathrm{e}^{-x/b} + D$$
for some constants $C > 0$ and $D \in \mathbb{R}$
- If $a = 1$, integrating $u'(x) = C(x + b)^{-1}$, get
$$u(x) = C \log(x + b) + D$$

Characterization of HARA utility

- Constants $C > 0$ and $D \in \mathbb{R}$ do not affect ordering of utility, so without loss of generality we may set $C = 1$ and $D = 0$
- In summary, all HARA utilities are

$$u(x) = \begin{cases} \frac{1}{a-1}(ax+b)^{1-1/a}, & (a \neq 0, 1) \\ -b\mathrm{e}^{-x/b}, & (a = 0) \\ \log(x+b), & (a = 1) \end{cases}$$

  where x is in the range $ax + b > 0$
- Some special cases:
  - CES utility corresponds to $a \neq 0, 1$ and $b = 0$
  - Cobb-Douglas utility corresponds to $a = 1$ and $b = 0$
  - Quadratic utility corresponds to $a = -1$

Aggregation with HARA utilities

- Let us show that when agents have HARA utilities with identical coefficient a, then computation of equilibrium is straightforward
- I agents, indexed by $i = 1, \dots, I$
- One consumption good, S states of the world indexed by $s = 1, \dots, S$
- Probability of state s is $\pi_s > 0$
- Agent i's endowment $e_i = (e_{i1}, \dots, e_{iS})$
- Aggregate endowment $e = \sum_{i=1}^{I} e_i$; with slight abuse of notation, write $e = (e_1, \dots, e_S)$
- Agent i has HARA utility

$$u_i(x) = \frac{1}{a-1}(ax + b_i)^{1-1/a},$$

so a is common across agents but b_i arbitrary

Aggregation with HARA utilities

Proposition

Let $b = \sum_{i=1}^{I} b_i$. Then equilibrium price is determined as if the economy consists of one (representative) agent with HARA utility with parameter (a, b) and initial endowment $e = (e_1, \dots, e_S)$. The equilibrium price is

$$p_s = \begin{cases} \pi_s (a e_s + b)^{-1/a}, & (a \neq 0) \\ \pi_s \mathrm{e}^{-e_s/b}. & (a = 0) \end{cases}$$

The equilibrium allocation (x_i) satisfies

$$\begin{cases} a x_{is} + b_i = \lambda_i^{-a} (a e_s + b), & (a \neq 0) \\ \frac{x_{is}}{b_i} = -\log \lambda_i + \frac{e_s}{b}, & (a = 0) \end{cases}$$

where $\lambda_i > 0$ is agent i's Lagrange multiplier that is determined from the budget constraint.

Proof

- Assume $a \neq 0$ (the case $a = 0$ is similar)
- Lagrangian of agent i is

$$L_i = \sum_{s=1}^{S} \pi_s u_i(x_s) + \lambda_i(p \cdot e_i - p \cdot x)$$

- First-order condition with respect to x_s is

$$\lambda_i p_s = \pi_s u_i'(x_{is}) = \pi_s (a x_{is} + b_i)^{-1/a}$$

- Hence

$$\begin{aligned} & \frac{\pi_s}{p_s}(a x_{is} + b_i)^{-1/a} = \frac{\pi_1}{p_1}(a x_{i1} + b_i)^{-1/a} \\ \iff & \left(\frac{\pi_s}{p_s}\right)^{-a}(a x_{is} + b_i) = \left(\frac{\pi_1}{p_1}\right)^{-a}(a x_{i1} + b_i). \end{aligned}$$

Proof

- Adding across i and using $b = \sum_{i=1}^{I} b_i$ and market clearing $\sum_{i=1}^{I} x_{is} = \sum_{i=1}^{I} e_{is} = e_s$, get

$$\left(\frac{\pi_s}{p_s}\right)^{-a}(ae_s + b) = \left(\frac{\pi_1}{p_1}\right)^{-a}(ae_1 + b)$$
$$\iff \frac{\pi_s}{p_s}(ae_s + b)^{-1/a} = \frac{\pi_1}{p_1}(ae_1 + b)^{-1/a}$$

- Hence quantity $\frac{\pi_s}{p_s}(ae_s + b)^{-1/a}$ independent of s; since price level arbitrary, set

$$\frac{\pi_s}{p_s}(ae_s + b)^{-1/a} = 1 \iff p_s = \pi_s(ae_s + b)^{-1/a}$$

- Above equation is exactly FOC of representative agent with HARA (a, b) utility consuming aggregate endowment (and Lagrange multiplier $\lambda = 1$)

Proof

- To pin down equilibrium allocation, recall FOC

$$\lambda_i p_s = \pi_s u_i'(x_{is}) = \pi_s(ax_{is} + b_i)^{-1/a}$$

- Since $p_s = \pi_s(ae_s + b)^{-1/a}$, get

$$\begin{aligned} & \lambda_i \pi_s(ae_s + b)^{-1/a} = \pi_s(ax_{is} + b_i)^{-1/a} \\ \iff & ax_{is} + b_i = \lambda_i^{-a}(ae_s + b) \end{aligned}$$

- Can pin down λ_i by using budget constraint

$$\sum_{s=1}^{S} p_s x_{is} = \sum_{s=1}^{S} p_s e_{is},$$

because p_s already determined and x_{is} expressed using λ_i only

Some observations

Looking at proof technique,

- Get same conclusion if utility is state-dependent such that

$$u_i(x_s) = \frac{1}{a-1}(ax_{is} + b_{is})^{1-1/a}$$

  in state s

- Important that a is common across agents
- Important that π_s is common across agents (objective probability or identical beliefs)

Chapter VII

International trade

Ricardo's model

- Ricardo's model is GE model of international trade
- Production economy
- Only input is labor, which is immobile across countries (no immigration)
- Production technology is linear in labor
- Outputs freely traded across countries

Numerical example of Ricardo's model

- Two countries ($i = A, B$), two consumption goods ($l = 1, 2$)
- One (representative) agent in each country
- Labor endowments: $(e_A, e_B) = (1, 2)$ (B is large country)
- Linear technology: if employ labor e, output is $y = ae$
- Example: $(a_{A1}, a_{A2}, a_{B1}, a_{B2}) = (10, 5, 4, 1)$, where a_{il}: productivity of country i to produce good l (think of good 1 as rice and good 2 as electric vehicle)
- For simplicity, assume all agents have utility $u(x_1, x_2) = x_1 x_2$, which is monotonic transformation of Cobb-Douglas utility

$$\frac{1}{2}\log x_1 + \frac{1}{2}\log x_2$$

Equilibrium

- This is an example of production economy
- Therefore competitive equilibrium defined by
  1. Utility maximization
  2. Profit maximization
  3. Market clearing
- Need to be clear about how goods are traded, autarky (no trade) or international trade?

Autarky equilibrium in country A: utility maximization

- Consider country A in isolation (autarky)
- Number of goods is 3, because there are rice ($l = 1$), electric vehicles ($l = 2$), and labor
- Let $p_1 = 1$ (price of rice), $p_2 = p$ (price of electric vehicle), and wage (price of labor) w
- Budget constraint of agents (workers):

$$x_1 + px_2 \leq we_A$$

- Since $e_A = 1$, using Cobb-Douglas formula, demand is

$$(x_1, x_2) = \left(\frac{w}{2}, \frac{w}{2p}\right)$$

Autarky equilibrium in country A: profit maximization

- A firm producing good l seeks to maximize profit $p_l y - we$, where y: output and e: labor input
- Since technology linear ($y = a_l e$), profit is

$$p_l y - we = p_l a_l e - we = (p_l a_l - w)e$$

- Hence profit linear in labor input e; optimal choice is

$$e = \begin{cases} \infty, & (p_l a_l > w) \\ \text{arbitrary}, & (p_l a_l = w) \\ 0, & (p_l a_l < w) \end{cases}$$

- Note: if $p_l a_l = w$, then profit is zero

Autarky equilibrium in country A: market clearing

- In equilibrium, all goods must be produced
- Hence profit maximization implies $p_l a_l = w$
  - Good 1: $w = p_1 a_{A1} = 1 \cdot 10 = 10$
  - Good 2: $w = p_2 a_{A2} = p \cdot 5 \iff p = 2$
- With these prices and wage, demand is

$$(x_1, x_2) = \left(\frac{w}{2}, \frac{w}{2p}\right) = \left(5, \frac{5}{2}\right)$$

- Market clearing:
  - Good 1: $x_1 = y_1 = a_1 e_1 \iff e_1 = x_1/a_1 = 5/10 = 1/2$
  - Good 2: $x_2 = y_2 = a_2 e_2 \iff e_2 = x_2/a_2 = (5/2)/5 = 1/2$
  - Labor market clears because $e_1 + e_2 = 1 = e_A$

Autarky equilibrium in country A

- We have now verified all equilibrium conditions
- Price and wage are $(p_1, p_2) = (1, 2)$ and $w = 10$
- Consumption is $(x_1, x_2) = (5, 5/2)$
- Labor input is $(e_1, e_2) = (1/2, 1/2)$
- Utility of agent is $U_A^a = x_1 x_2 = 25/2$

Autarky equilibrium in country B

- Let price be $(p_1, p_2) = (1, p)$ and wage w
- Zero profit conditions are
  - Good 1: $w = p_1 a_1 = 1 \cdot 4 = 4$
  - Good 2: $w = p_2 a_2 = p \cdot 1 \iff p = 4$
- With these prices and wage, demand is

$$(x_1, x_2) = \left(\frac{we_B}{2}, \frac{we_B}{2p}\right) = (4, 1)$$

- Market clearing:
  - Good 1: $x_1 = y_1 = a_1 e_1 \iff e_1 = x_1/a_1 = 4/4 = 1$
  - Good 2: $x_2 = y_2 = a_2 e_2 \iff e_2 = x_2/a_2 = 1/1 = 1$
  - Labor market clears because $e_1 + e_2 = 2 = e_B$
- Utility of agent is $U_B^a = x_1 x_2 = 4$

Free trade equilibrium

- Now let's solve for the free trade equilibrium and compare to autarky
- Free trade means that labor stays within the border (no immigration allowed, otherwise it's same as one country) but goods are freely traded across border
- What will happen under free trade?
  - Since A's wage ($w_A = 10$) much higher than B ($w_B = 4$), would A's domestic jobs be lost to B?
  - Since A's industries much more productive, would B's industries be wiped out?
- Both predictions incorrect: need formal analysis

Free trade equilibrium: country A

- Since A much more productive overall, assume A produces both goods in equilibrium
- Hence profit maximization implies $p_l a_l = w$
  - Good 1: $w_A = p_1 a_{A1} = 1 \cdot 10 = 10$
  - Good 2: $w_A = p_2 a_{A2} = p \cdot 5 \iff p = 2$
- With these prices and wage, demand is

$$(x_{A1}, x_{A2}) = \left(\frac{w_A e_A}{2}, \frac{w_A e_A}{2p} \right) = \left(5, \frac{5}{2} \right)$$

Free trade equilibrium: country B

- Good price is $(p_1, p_2) = (1, 2)$
- Letting w_B be wage, profit maximization requires $p_l a_{Bl} \le w_B$, with equality if positive amount of good produced
- Since $p_1 a_{B1} = 4 > 2 = p_2 a_{B2}$, it must be $4 = p_1 a_{B1} = w_B$
- With these prices and wage, demand is

$$(x_{B1}, x_{B2}) = \left(\frac{w_B e_B}{2}, \frac{w_B e_B}{2p}\right) = (4, 2)$$

- Firm 2 not profitable, so labor input is $e_{B1} = e_B = 2$ and $e_{B2} = 0$ (all workers employed by firm 1)

Free trade equilibrium: market clearing

- Since firm 2 in country B does not operate, firm 2 in country A must produce all global supply of good 2
- Hence market clearing is

$$x_{A2} + x_{B2} = a_{A2} e_{A2} \iff \frac{5}{2} + 2 = 5 e_{A2} \iff e_{A2} = \frac{9}{10}$$

- Remaining worker $e_{A1} = e_A - e_{A2} = 1/10$ work in firm 1
- Market clearing for good 1?

$$\begin{aligned} & x_{A1} + x_{B1} = a_{A1} e_{A1} + a_{B1} e_{B1} \\ \iff & 5 + 4 = 10 \cdot \frac{1}{10} + 4 \cdot 2 \\ \iff & 9 = 9, \end{aligned}$$

so OK

Comparing autarky to free trade

Regime	Autarky	Free trade
Consumption	$A : (5, 5/2),\ B : (4, 1)$	$A : (5, 5/2),\ B : (4, 2)$
Labor	$A : (1/2, 1/2),\ B : (1, 1)$	$A : (1/10, 9/10),\ B : (2, 0)$
Utility	$U_A^a = 25/2,\ U_B^a = 4$	$U_A^f = 25/2,\ U_B^f = 8$

- So B specializes in good 1 (rice) and utility goes up
- Workers in A shift to producing good 2 (electric vehicles) but utility unchanged
- In general, small or inefficient country tends to gain from trade because price change allows to reoptimize

Ricardo's model: general formulation

- Two countries (A, B), L goods $(l = 1, \dots, L)$
- Productivity of country i in sector l is a_{il}
- Labor endowment e_A, e_B
- Utility function can be general: u_A, u_B
- Equilibrium defined by
  1. Utility maximization
  2. Profit maximization
  3. Market clearing

Comparative advantage

- In previous example, we solved for equilibrium by guessing, but there is general way
- Define *comparative advantage* of country A over B for producing good l to be a_{Al}/a_{Bl}
- By relabeling goods if necessary, may assume

$$\frac{a_{A1}}{a_{B1}} > \cdots > \frac{a_{AL}}{a_{BL}},$$

so comparative advantage is decreasing in label of good

In equilibrium, countries specialize

Proposition

In equilibrium, there exists a good l^* such that all goods $l < l^*$ are produced by A only, and all goods $l > l^*$ are produced by B only

- To prove this, define

$$l^* = \max_l \{\text{good } l \text{ is produced by country } A\}$$

- By definition, all goods $l > l^*$ are produced by B only
- Consider a good $l \leq l^*$; profit maximization (hence zero profit) implies
  - $p_l a_{Al} \leq w_A$, $=$ if good l produced
  - $p_l a_{Bl} \leq w_B$, $=$ if good l produced
  - By assumption A produces l^*, so $p_{l^*} a_{Al^*} = w_A$

In equilibrium, countries specialize

- Suppose to the contrary that B produces a good $l < l^*$
- Then by above argument $p_l a_{Bl} = w_B \ge p_{l^*} a_{Bl^*}$
- Hence $a_{Bl}/a_{Bl^*} \ge p_{l^*}/p_l$
- Similarly, $p_l a_{Al} \le w_A = p_{l^*} a_{Al^*}$ implies $a_{Al}/a_{Al^*} \le p_{l^*}/p_l$
- Hence
$$\frac{a_{Al}}{a_{Al^*}} \le \frac{p_{l^*}}{p_l} \le \frac{a_{Bl}}{a_{Bl^*}} \implies \frac{a_{Al}}{a_{Bl}} \le \frac{a_{Al^*}}{a_{Bl^*}},$$
contradicting definition of comparative advantage

Solving for equilibrium in Ricardo's model

- We can solve for equilibrium in general model as follows
  1. Relabel goods so that comparative advantage is decreasing in l
  2. Guess l^*, the largest l country A produces
  3. Set $p_{l^*} = 1$ (normalization); then zero profit $p_{l^*} a_{Al^*} = w_A$ implies $w_A = a_{Al^*}$
  4. All goods $l < l^*$ must be produced by A, so zero profit condition $p_l a_{Al} = w_A$ implies $p_l = a_{Al^*}/a_{Al}$
  5. Assume l^* also produced by B; then zero profit implies $p_{l^*} a_{Bl^*} = w_B$ and hence $w_B = a_{Bl^*}$
  6. All goods $l > l^*$ must be produced by B, so zero profit condition $p_l a_{Bl} = w_B$ implies $p_l = a_{Bl^*}/a_{Bl}$

Solving for equilibrium in Ricardo's model

- Now all prices $p_1, \dots, p_L$ determined
- Compute demand x_{Al} and x_{Bl} by solving UMP
- Now we use market clearing:
  1. Good $l < l^*$ produced only by A, hence
  $$x_{Al} + x_{Bl} = a_{Al} e_{Al} \iff e_{Al} = \frac{x_{Al} + x_{Bl}}{a_{Al}}$$
  2. Good $l > l^*$ produced only by B, hence
  $$x_{Al} + x_{Bl} = a_{Bl} e_{Bl} \iff e_{Bl} = \frac{x_{Al} + x_{Bl}}{a_{Bl}}$$
  3. Compute remaining labor (employed in l^* sector) as
  $$e_{Al^*} = e_A - \sum_{l<l^*} e_{Al} \text{ and } e_{Bl^*} = e_B - \sum_{l>l^*} e_{Bl}$$
  4. If both nonnegative, that's equilibrium; if not, guess of l^* is incorrect, so move it down if $e_{Al^*} < 0$ and up if $e_{Bl^*} < 0$

Free trade in small open economies

- Ricardo's model (numerical example) suggest that free trade is Pareto improving
- But that is not necessarily the case, as following example shows
- Consider a small country with two agents and two goods, where all agents have utility $u(x_1, x_2) = x_1 x_2$
- Initial endowment $e_1 = (9, 1)$ and $e_2 = (1, 9)$
- Autarky equilibrium clearly $p = (p_1, p_2) = (1, 1)$, $x_1 = (x_{11}, x_{12}) = (5, 5)$, and $x_2 = (x_{21}, x_{22}) = (5, 5)$, with utility $5 \cdot 5 = 25$ for both agents

Free trade in small open economies

- Suppose country moves to free trade, and suppose international price is $p = (p_1, p_2) = (1, 2)$
- Under new price, agent 1's demand is

$$(x_{11}, x_{12}) = \left(\frac{1 \cdot 9 + 2 \cdot 1}{2 \cdot 1}, \frac{1 \cdot 9 + 2 \cdot 1}{2 \cdot 2} \right) = (11/2, 11/4),$$

  with utility $(11/2)(11/4) = 121/8 < 25$
- Under new price, agent 2's demand is

$$(x_{21}, x_{22}) = \left(\frac{1 \cdot 1 + 2 \cdot 9}{2 \cdot 1}, \frac{1 \cdot 1 + 2 \cdot 9}{2 \cdot 2} \right) = (19/2, 19/4),$$

  with utility $(19/2)(19/4) = 361/8 > 25$
- So agent 1 worse off in free trade, agent 2 better off

Free trade not necessarily Pareto improving

- The reason why free trade is not necessarily Pareto improving is because people have different endowments
- Example: price of rice in California about half in Japan (for same variety)
- Hence if there is trade liberalization in rice, Japanese rice farmers worse off (due to more competition and cheaper price) and Californian rice farmers better off (due to more demand)
- Hence many governments concerned with trade, and may come up with all sorts of policies (e.g., tariff, import quota, etc.)

How to make free trade Pareto improving

- But there is simple way to make free trade Pareto improving
- Idea: introduce direct tax/subsidies to make previous consumption just affordable
- In previous example, autarky equilibrium was $x_1^a = x_2^a = (5,5)$
- So define t_i to make autarky allocation just affordable
  - Agent 1: $p \cdot x_1^a = p \cdot e_1 - t_1 \iff t_1 = p \cdot (e_1 - x_1^a) = (1,2) \cdot (4,-4) = -4$,
  - Agent 2: $p \cdot x_2^a = p \cdot e_2 - t_2 \iff t_2 = p \cdot (e_2 - x_2^a) = (1,2) \cdot (-4,4) = 4$,
- After transfer, both agents have wealth $w_i = p \cdot x_i^a = 15$, so demand is

$$x_i^f = \left(\frac{15}{2 \cdot 1}, \frac{15}{2 \cdot 2}\right) = (15/2, 15/4)$$

- Utility $(15/2)(15/4) = 225/8 > 25$, so both agents better off under free trade

General formulation

Theorem

Consider a country with economy $\mathcal{E} = \{I, (e_i), (u_i)\}$. Suppose the country is small and takes world price p as given. Then there exist transfer payments (t_i) such that the free trade allocation weakly Pareto dominates the autarky equilibrium allocation. More precisely, if x_i^f solves

$$\begin{aligned} &\text{maximize} && u_i(x) \\ &\text{subject to} && p \cdot x \leq p \cdot e_i - t_i, \end{aligned}$$

then the free trade allocation (x_i^f) weakly Pareto dominates the autarky equilibrium allocation (x_i^a).

Proof

- By market clearing, autarky equilibrium allocation (x_i^a) satisfies $\sum_{i=1}^I x_i^a \le \sum_{i=1}^I e_i$
- For each i, choose y_i such that $y_i \ge x_i^a$ and $\sum_{i=1}^I y_i = \sum_{i=1}^I e_i$
- Define transfer t_i so that y_i is just affordable at world price, so $p \cdot y_i = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - y_i)$
- By definition of y_i, we have

$$\sum_{i=1}^I t_i = \sum_{i=1}^I p \cdot (e_i - y_i) = p \cdot \sum_{i=1}^I (e_i - y_i) = 0,$$

  so government budget balanced
- Since $x_i^a \le y_i$, we have $p \cdot x_i^a \le p \cdot y_i = p \cdot e_i - t_i$, so x_i^a affordable under new budget constraint, hence $u_i(x_i^f) \ge u_i(x_i^a)$

Policy implication

- For small countries, free trade always better than autarky if appropriate tax/transfer implemented
- Actually can show that free trade always better than any trade policy if appropriate tax/transfer implemented
- See end-of-chapter exercise

Free trade in general equilibrium

- We showed free trade is great for small countries, and proof is easy because price exogenous (partial equilibrium)
- Not obvious if conclusion holds in general equilibrium
- Suppose global economy is $\mathcal{E} = \{I, (e_i), (u_i)\}$
- Countries indexed by $c = 1, \dots, C$
- Set of residents of country c denoted by $I_c \subset I$
- Suppose countries initially in autarky; can we show free trade is Pareto improving after appropriate tax/transfer within each country?

Free trade in general equilibrium

Theorem (Efficiency of free trade with transfers)

Let $\mathcal{E} = \{I, (e_i), (u_i)\}$ be the global economy, where u_i is continuous, quasi-concave, and locally nonsatiated. Then there exist a price vector p, an allocation (x_i^f), and transfer payments (t_i) such that

1. $(p, (x_i^f), (t_i))$ *is a free trade equilibrium with transfer payments,*
2. *for each country c, transfer payments are budget-feasible, so* $\sum_{i \in I_c} t_i = 0$,
3. *the free trade allocation (x_i^f) weakly Pareto dominates the autarky allocation (x_i^a), that is, $u_i(x_i^f) \geq u_i(x_i^a)$ for all i, and*
4. *the free trade allocation (x_i^f) is Pareto efficient.*

Proof

- Unlike partial equilibrium, difficulty is that we are not given global price
- Need to construct price, allocation, and transfers cleverly
- Let $\{p_c, (x_i^a)_{i \in I_c}\}$ be autarky equilibrium in country c
- By market clearing, $\sum_{i \in I_c} x_i^a \le \sum_{i \in I_c} e_i$
- Choose y_i such that $y_i \ge x_i^a$ and $\sum_{i \in I_c} y_i = \sum_{i \in I_c} e_i$
- Consider hypothetical global economy $\mathcal{E}' = \{I, (y_i), (u_i)\}$, so agent i has initial endowment y_i (actual economy is e_i)

Proof

- Let $\{p, (x_i^f)\}$ be equilibrium of $\mathcal{E}'$
- By first welfare theorem, (x_i^f) is Pareto efficient
- To support it as equilibrium with transfer payments, define t_i to make y_i just affordable in actual economy, so $p \cdot y_i = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - y_i)$
- As in partial equilibrium case, we have $\sum_{i \in I_c} t_i = 0$, so government budget balances within each country
- x_i^f is demand with initial endowment y_i, so $p \cdot x_i^f \le p \cdot y_i = p \cdot e_i - t_i$; hence x_i^f is demand with initial endowment e_i and transfer t_i
- Since $x_i^a \le y_i$, we have $p \cdot x_i^a \le p \cdot y_i = p \cdot e_i - t_i$, so x_i^a affordable, hence $u_i(x_i^f) \ge u_i(x_i^a)$

Example

- Three agents ($i = 1, 2, 3$), two goods ($l = 1, 2$), two countries $c = A, B$)
- Agents 1 and 2 residents of A, agent 3 resident of B
- Utility functions

$$\begin{aligned}
u_1(x_1, x_2) &= x_1^2 x_2, \\
u_2(x_1, x_2) &= x_1 x_2^2, \\
u_3(x_1, x_2) &= x_1 x_2.
\end{aligned}$$

- Endowments $e_1 = e_2 = (3, 3)$ and $e_3 = (22, 8)$
- Compute autarky equilibrium and free trade equilibrium (with or without transfers)

Autarky equilibrium in country A

- Let prices be $p_1 = 1$ and $p_2 = p$
- Using Cobb-Douglas formula, demand is

$$(x_{11}, x_{12}) = \left(\frac{2}{3}(3+3p), \frac{1}{3p}(3+3p)\right) = \left(2+2p, \frac{1+p}{p}\right),$$
$$(x_{21}, x_{22}) = \left(\frac{1}{3}(3+3p), \frac{2}{3p}(3+3p)\right) = \left(1+p, \frac{2+2p}{p}\right)$$

- Market clearing for good 1:

$$(2+2p) + (1+p) = 3+3 \iff p = 1$$

- Equilibrium allocation $x_1 = (4,2)$, $x_2 = (2,4)$, utility $U_1 = U_2 = 4^2 \times 2 = 32$

Free trade equilibrium without transfer

- In free trade, agent 3's demand is

$$(x_{31}, x_{32}) = \left(\frac{1}{2}(22+8p), \frac{1}{2p}(22+8p)\right) = \left(11+4p, \frac{11+4p}{p}\right)$$

- Market clearing for good 1:

$$(2+2p)+(1+p)+(11+4p) = 3+3+22 \iff p = 2$$

- Equilibrium allocation $x_1 = (6, 3/2)$, $x_2 = (3, 3)$, $x_3 = (19, 19/2)$, utility $U_1 = 54$, $U_2 = 27$, $U_3 = 361/2$
- So agent 1 better off and agent 2 worse of in free trade

Free trade with transfer

- Let's make a Pareto improvement by introducing tax/subsidy in country A
- Start with autarky equilibrium allocation $x_1^a = (4, 2)$, $x_2^3 = (2, 4)$, $x_3^a = (22, 8)$
- New demand is

$$
\begin{aligned}
(x_{11}, x_{12}) &= \left(\frac{2}{3}(4+2p), \frac{1}{3p}(4+2p)\right), \\
(x_{21}, x_{22}) &= \left(\frac{1}{3}(2+4p), \frac{2}{3p}(2+4p)\right), \\
(x_{31}, x_{32}) &= \left(\frac{1}{2}(22+8p), \frac{1}{2p}(22+8p)\right) = \left(11+4p, \frac{11+4p}{p}\right)
\end{aligned}
$$

Free trade with transfer

- Market clearing for good 1:

$$\frac{2(4+2p)}{3}+\frac{2+4p}{3}+(11+4p)=4+2+22 \iff p=\frac{41}{20}$$

- To support this price as free trade equilibrium, transfer t_i must satisfy $p \cdot x_i^a = p \cdot e_i - t_i \iff t_i = p \cdot (e_i - x_i^a)$
- Hence

$$t_1 = (1, 41/20) \cdot [(3,3)-(4,2)] = \frac{21}{20},$$
$$t_2 = (1, 41/20) \cdot [(3,3)-(2,4)] = -\frac{21}{20}$$

- Can compute x_i^f and U_i^f for all i and check $U_i^f > U_i^a$, though algebra is quite tedious

Transportation cost

- So far we assumed goods can be transported without cost
- But in reality it is costly to transport goods
- Actually trade model with transportation cost can be thought of as general equilibrium model with production by distinguishing goods by location
- Example: we can produce a banana in U.S. (one output) from a banana in Ecuador and shipping service (two inputs)

Illustration of transportation cost

- Two countries ($i = A, B$), two physical goods (apple and banana)
- Utility $u(x_1, x_2) = \frac{1}{2} \log x_1 + \frac{1}{2} \log x_2$
- Endowment $e_A = (3, 1)$ and $e_B = (1, 3)$
- Suppose 20% of goods perish when shipped to other country, so production technology $y = \frac{4}{5}x$, where x: export and y: import
- To compute equilibrium, note there are 4 goods (apple and banana in A, B); guess which country export and import which good, and use utility maximization, profit maximization, market clearing (can be tedious but nothing complicated)

Free trade equilibrium

- Since A has lots of good 1 (apple), can guess A will export apples and import bananas
- Let $p^A = (p_1^A, p_2^A) = (1, p)$ be price vector in country A
- By symmetry, guess $p^B = (p, 1)$
- If apple exporter in A exports x, profit is

$$p \cdot \frac{4}{5}x - 1 \cdot x = \left(\frac{4}{5}p - 1\right)x$$

- Since profit linear in x, it must be zero; hence $p = 5/4$

Free trade equilibrium

- With endowment $e_A = (3, 1)$, demand in A is

$$x_A = \left(\frac{1}{2}(3+p), \frac{1}{2p}(3+p)\right) = (17/8, 17/10)$$

- Symmetric argument shows demand in B is $x_B = (17/10, 17/8)$
- A exports $3 - 17/8 = 7/8$ apples, which become $7/10$ apples in B by the time it reaches destination
- B exports $3 - 17/8 = 7/8$ bananas, which become $7/10$ bananas in A by the time it reaches destination
- If countries not symmetric, argument more complicated but idea is same

Chapter VIII

Finance

Finance

- A general equilibrium model becomes a model of finance when goods are distinguished by states of the world
- We first study no-arbitrage asset pricing, which is useful for computing prices of some assets given prices of other assets (e.g., option pricing)
- By studying a general equilibrium model with HARA or quadratic utility, we can derive the capital asset pricing model (CAPM)

No-arbitrage asset pricing

- Consider a two period model with time denoted by $t = 0, 1$ (today and tomorrow)
- At $t = 1$, the economy will be in one of the states denoted by $s = 1, \dots, S$
- Only one physical good
- Assets are indexed by $j = 1, \dots, J$
  - Asset j trades at price q_j at $t = 0$
  - Asset j pays A_{sj} (if $A_{sj} < 0$, then asset holder must deliver $-A_{sj} > 0$)
- Convenient to treat $t = 0$ (today) as a new state denoted by $s = 0$

Payoff matrix

- Define
  - asset price vector $q = (q_1, \dots, q_J)$
  - payoff matrix $A = (A_{sj})$ (which is $S \times J$ matrix)
- Since investor must pay q_j to acquire one share of asset j, matrix of net payoffs (including $s = 0$) is

$$W := \begin{bmatrix} -q' \\ A \end{bmatrix} = \begin{bmatrix} -q_1 & \cdots & -q_J \\ A_{11} & \cdots & A_{1J} \\ \vdots & \ddots & \vdots \\ A_{S1} & \cdots & A_{SJ} \end{bmatrix}$$

Asset span

- If investor holds θ_j shares of asset j and $\theta = (\theta_1, \dots, \theta_J)$ is portfolio, then portfolio payoff is $W\theta$
- To see this,
  - for $s = 0$, holding portfolio θ costs
    $$\sum_{j=1}^{J} q_j \theta_j = q'\theta,$$
    so payoff is $-q'\theta = (W\theta)_0$
  - for $s = 1, \dots, S$, portfolio pays
    $$\sum_{j=1}^{J} A_{sj} \theta_j = (W\theta)_s$$
- If portfolio choice unrestricted (including short sales), set of possible payoffs (asset span) is the vector space
  $$\langle W \rangle := \left\{ W\theta : \theta \in \mathbb{R}^J \right\} \subset \mathbb{R}^{1+S}$$

Arbitrage

- Suppose investors like the physical good (strongly monotonic preferences)
- If there exists portfolio $\theta \in \mathbb{R}^J$ such that $W\theta > 0$ (meaning $W\theta \in \mathbb{R}^{1+S}_+ \setminus \{0\}$), then investors can consume arbitrarily large amount of one good without paying anything (free lunch; arbitrage), violating equilibrium
- We say that asset span $\langle W \rangle$ exhibits no arbitrage if $\langle W \rangle \cap \mathbb{R}^{1+S}_+ = \{0\}$
- What can we say about asset prices if there is no arbitrage?

Fundamental Theorem of Asset Pricing

Theorem

The asset span $\langle W \rangle$ *exhibits no-arbitrage if and only if there exists* $p \in \mathbb{R}^S_{++}$ *such that* $[1, p']W = 0$.
In this case, the asset prices are given by

$$q_j = \sum_{s=1}^{S} p_s A_{sj}.$$

$p_s > 0$ *is called the* *state price* *in state* s.

Fundamental Theorem of Asset Pricing

Theorem

The asset span $\langle W \rangle$ exhibits no-arbitrage if and only if there exists $p \in \mathbb{R}^S_{++}$ such that $[1, p']W = 0$.
In this case, the asset prices are given by

$$q_j = \sum_{s=1}^{S} p_s A_{sj}.$$

$p_s > 0$ is called the state price in state s.

- Absence of arbitrage implies that we can put a price on the good delivered in state s only
- Asset price can be computed as $\sum$(state price) $\times$ (delivery)

Proof of "if" part

- Suppose there exists $p \in \mathbb{R}^S_{++}$ such that $[1, p']W = 0$
- By definition of W, get

$$0 = [1, p']W = [1, p'] \begin{bmatrix} -q' \\ A \end{bmatrix} = -q' + p'A \iff q' = p'A$$

- Comparing j-th entry, get $q_j = \sum_{s=1}^S p_s A_{sj}$
- Suppose to the contrary that there exists an arbitrage, so there exists $w \in \langle W \rangle$ such that $w > 0$
- Then

$$0 < w_0 + \sum_{s=1}^S p_s w_s = [1, p']w = [1, p']W\theta = 0$$

for some portfolio θ, which is contradiction

Proof of "only if" part

- Suppose $\langle W\rangle$ exhibits no arbitrage, so $\langle W\rangle \cap \mathbb{R}^{1+S}_+ = \{0\}$
- Idea: use strong version of separating hyperplane theorem
- Define the unit simplex

$$\Delta := \left\{ w \in \mathbb{R}^{1+S}_+ : \sum_{s=0}^{S} w_s = 1 \right\}$$

- Clearly $\Delta \subset \mathbb{R}^{1+S}_+ \setminus \{0\}$
- Since $\langle W\rangle \cap \mathbb{R}^{1+S}_+ = \{0\}$, get $\langle W\rangle \cap \Delta = \emptyset$

Proof of "only if" part

- $\langle W\rangle$ is a finite-dimensional vector space, which is nonempty, closed (intuitive but hard to prove), and convex
- Δ is clearly nonempty, compact, and convex
- Since $\langle W\rangle \cap \Delta = \emptyset$, we can apply strong version of separating hyperplane theorem
- Hence there exists nonzero vector $\lambda \in \mathbb{R}^{1+S}$ such that

$$\sup_{w\in\langle W\rangle} \lambda \cdot w < \inf_{d\in\Delta} \lambda \cdot d$$

- Using definition of asset span $\langle W\rangle$, we get

$$\sup_{\theta\in\mathbb{R}^J} \lambda' W\theta < \inf_{d\in\Delta} \lambda \cdot d$$

Proof of "only if" part

- Using definition of asset span $\langle W \rangle$, we get

$$\sup_{\theta \in \mathbb{R}^J} \lambda' W \theta < \inf_{d \in \Delta} \lambda \cdot d$$

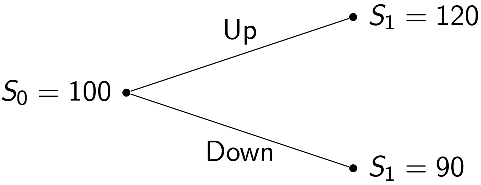
- Then it must be $\lambda' W = 0$
  - To see this, suppose $v := \lambda' W \neq 0$; let $v = (v_1, \dots, v_J)$ and suppose $v_j \neq 0$
  - Since θ arbitrary, choose $\theta = (0, \dots, \alpha v_j, \dots, 0)$
  - Then $\lambda' W \theta = \alpha v_j^2 \to \infty$ as $\alpha \to \infty$, which would exceed $\inf_{d \in \Delta} \lambda \cdot d$ eventually
  - Hence $v_j = 0$ for all j, and $\lambda' W = 0$
- $\lambda' W = 0$ implies $0 < \inf_{d \in \Delta} \lambda' d$; in particular, setting $d_s = 1$ for one s and $d_{s'} = 0$ for all others, get $\lambda_s > 0$
- Letting $p_s = \lambda_s / \lambda_0$ and dividing $\lambda' W = 0$ by $\lambda_0 > 0$, get $[1, p'] W = 0$

Pricing of options by no-arbitrage

- Let us apply no-arbitrage asset pricing to price options
- Broadly speaking, there are two types of options
  - Call option: right (but no obligation) to buy stock at specified price
  - Put option: right (but no obligation) to sell stock at specified price
- "Specified price" is called strike price
- If holder of call option (with strike price K) exercise option when stock price is S, then payoff is $S - K$ (because pays K to buy stock that has value S in market)
- Similarly, payoff of exercising put option is $K - S$

Binomial option pricing: example

- Consider simple model with two dates $t = 0, 1$
- At $t = 0$, stock price is $S_0 = 100$; at $t = 1$, stock price is either $S_1 = 120$ or $S_1 = 90$
- Suppose interest rate is 10% between two dates
- What is call price with strike $K = 100$?



Binomial option pricing: example

- Consider an asset U that pays 1 in "Up" state and 0 otherwise; let p_u be its price at $t = 0$
- Consider an asset D that pays 1 in "Down" state and 0 otherwise; let p_d be its price at $t = 0$

Binomial option pricing: example

- Consider an asset U that pays 1 in "Up" state and 0 otherwise; let p_u be its price at $t = 0$
- Consider an asset D that pays 1 in "Down" state and 0 otherwise; let p_d be its price at $t = 0$
- A stock is the same as 120 shares of U and 90 shares of D
- Hence by no arbitrage, it must be

$$100 = S_0 = 120p_u + 90p_d$$

Binomial option pricing: example

- Similarly, a risk-free asset (that pays 1 no matter what) is the same as 1 share of U and 1 share of D
- Hence by no arbitrage, it must be

$$\frac{1}{1.1} = p_u + p_d$$

Binomial option pricing: example

- Similarly, a risk-free asset (that pays 1 no matter what) is the same as 1 share of U and 1 share of D
- Hence by no arbitrage, it must be

$$\frac{1}{1.1} = p_u + p_d$$

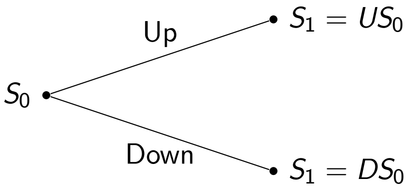
- We have two linear equations in two unknowns (p_u, p_d)
- After some algebra, solution is

$$(p_u, p_d) = \left(\frac{20}{33}, \frac{10}{33}\right)$$

- Call option pays $120 - 100 = 20$ in "Up" state, so it is the same as 20 shares of U; hence its price is $C = 20p_u = \frac{400}{33}$

Binomial option pricing: general two periods

- More generally, suppose stock price is S_0 at $t = 0$, and it is either $S_1 = US_0$ or $S_1 = DS_0$ at $t = 1$, where $U > D$
- Suppose gross risk-free rate R satisfies $U > R > D$



Binomial option pricing: general two periods

- By same argument as before,
  - Stock: $S_0 = p_u US_0 + p_d DS_0 \implies 1 = p_u U + p_d D$
  - Bond: $1/R = p_u + p_d \implies 1 = p_u R + p_d R$
- Taking difference, $0 = p_u(U - R) - p_d(R - D)$
- After some algebra, get

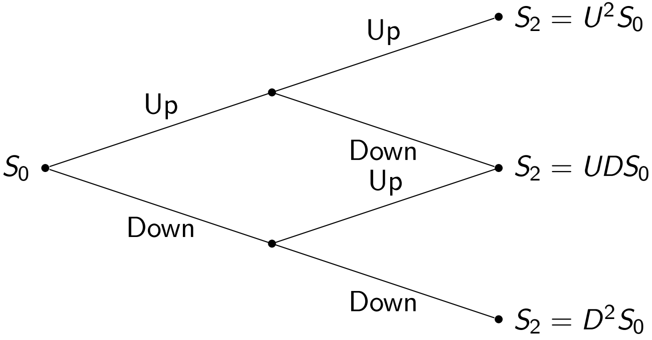
$$(p_u, p_d) = \frac{1}{R}(p, 1 - p),$$

  where $p = \frac{R-D}{U-D} \in (0, 1)$
- Can think of p as "risk-neutral" probability of "Up" state
- Call option price is then

$$C = p_u \max\{0, US_0 - K\} + p_d \max\{0, DS_0 - K\}$$

Binomial option pricing: general three periods

- More generally, suppose time is $t = 0, 1, 2$ and stock can grow either by factor U or D each period, where $U > D$
- Suppose gross risk-free rate R satisfies $U > R > D$



Risk-neutral pricing in three periods

- By same argument as before, state "2 Up" occurs with risk-neutral probability p^2
- State "1 Up 1 Down" occurs with risk-neutral probability $p(1-p)+(1-p)p=2p(1-p)$
- State "2 Down" occurs with risk-neutral probability $(1-p)^2$
- Hence (European) call option price is

$$C=\frac{1}{R^2}(p^2C_{uu}+2p(1-p)C_{ud}+(1-p)^2C_{dd}),$$

where $C_{uu}=\max\left\{0,U^2S_0-K\right\}$, $C_{ud}=\max\left\{0,UDS_0-K\right\}$, $C_{dd}=\max\left\{0,D^2S_0-K\right\}$

Binomial option pricing: general case

- Consider European call option with strike price K and expiration T
- Expected growth rate of stock price is μ and volatility is σ
- (Continuously compounded) risk-free rate is r
- What is price of European call option?

Binomial option pricing: general case

- To compute option price, we can divide time $0 \le t \le T$ into N subperiods, each with length $\Delta t = T/N$
- In each period, assume stock goes up or down, with factor

$$U = \mathrm{e}^{\mu \Delta t + \sigma \sqrt{\Delta t}}, \quad D = \mathrm{e}^{\mu \Delta t - \sigma \sqrt{\Delta t}}$$

- One period gross risk-free rate is $R = \mathrm{e}^{r \Delta t}$
- Define $p = \frac{R-D}{U-D} \in (0,1)$ be risk-neutral probability of "Up" state
- Then risk-neutral probability of "n Up, $N-n$ Down" is $\binom{N}{n} p^n (1-p)^{N-n}$, where $\binom{N}{n} = \frac{N!}{n!(N-n)!}$
- Therefore European call option price is

$$C = \underbrace{\frac{1}{R^N}}_{\text{discount}} \sum_{n=0}^{N} \underbrace{\binom{N}{n} p^n (1-p)^{N-n}}_{\text{risk-neutral probability}} \underbrace{\max\left\{0, U^n D^{N-n} S_0 - K\right\}}_{\text{terminal payoff}}$$

Binomial pricing of American options

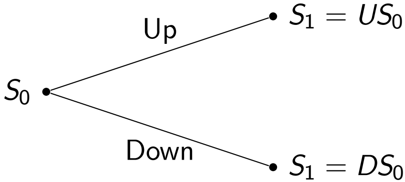
- We know early exercise of American call option is suboptimal
- Hence American call option price is same as European call option
- Not true for put options
- We can still apply binomial option pricing for American put options, but need a computer

Binomial pricing of American put options

- For simplicity, assume two periods $t = 0, 1$
- If exercise put option now, payoff is $K - S$
- Otherwise, at $t = 1$, put payoff is $\max\{0, K - S_1\}$
- Hence put price is

$$P = \max\left\{\underbrace{K - S_0}_{\text{Value if exercise}}, \underbrace{\frac{1}{R}(pP_u + (1-p)P_d)}_{\text{Value if wait}}\right\},$$

where $P_u = \max\{0, K - US_0\}$ and $P_d = \max\{0, K - DS_0\}$



Binomial pricing of American put options

- More generally, suppose you divide time $0 \le t \le T$ into N subperiods
- Let $P_{s,n}$ be value of put in s-th subperiod when stock went up n times
- If $s = N$ (terminal date), $P_{N,n} = \max\left\{K - U^n D^{N-n} S_0, 0\right\}$
- At subperiod s after n ups, put value is

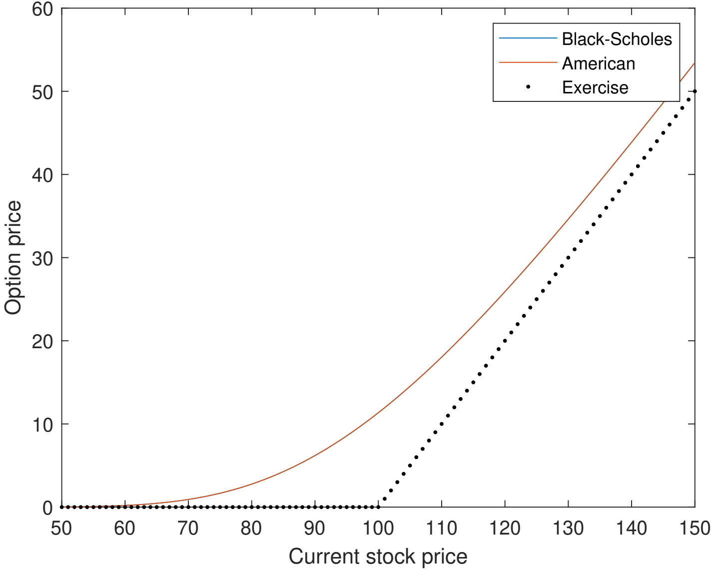
$$
\begin{aligned}
&P_{s,n} \\
&= \max\left\{\underbrace{K - U^n D^{s-n} S_0}_{\text{Value if exercise}}, \underbrace{\frac{1}{R}(pP_{s+1,n+1} + (1-p)P_{s+1,n})}_{\text{Value if wait}}\right\}
\end{aligned}
$$

- Starting from $\{P_{N,n}\}_{n=0}^{N}$, can compute any $P_{s,n}$ by iterating from backwards ($s = N, N-1, \dots, 2, 1, 0$)

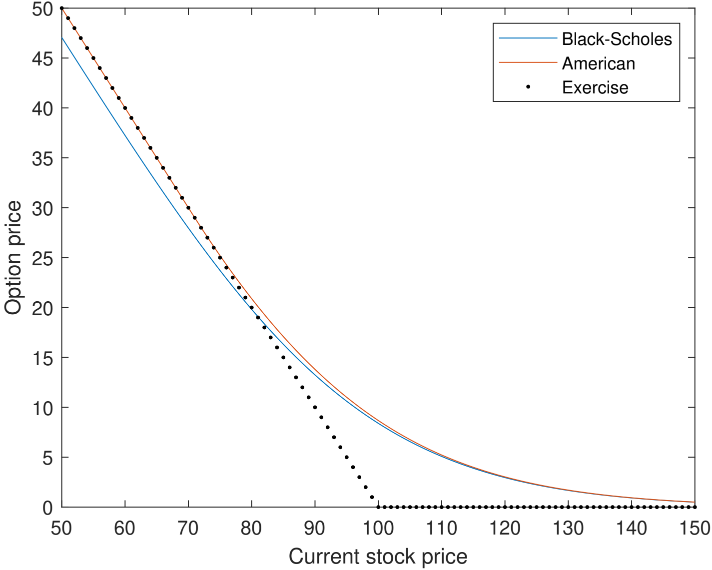
Example

- Consider American call and put options with expiration $T = 1$ year and strike price $K = 100$
- Suppose risk-free rate is 3%, expected stock return is 8%, and volatility is 25%
- Compute option prices using previous approach setting $r = 0.03$, $\mu = 0.08$, $\sigma = 0.25$, and using large N (say $N = 1000$)

American call option price



American put option price



Limitation of no-arbitrage asset pricing

- No-arbitrage asset pricing is useful for computing prices of derivatives (options) given fundamental assets (stocks)
- Its success is due to the weak assumptions: only optimizing behavior (absence of arbitrage) is required, and it is detail-independent (e.g., utility functions)
- However, weak assumptions imply weak conclusions: it does not say how prices of fundamental assets are determined in the first place

Capital Asset Pricing Model (CAPM)

- We embed no-arbitrage asset pricing into general equilibrium to derive CAPM
- I agents, indexed by $i = 1, \dots, I$
- Two period, with time denoted by $t = 0, 1$
- One consumption good, S states of the world at $t = 1$ indexed by $s = 0, 1, \dots, S$
- Probability of state s is $\pi_s > 0$

Capital Asset Pricing Model (CAPM)

- Agent i's endowment $e_i = (e_{i0}, e_{i1}, \dots, e_{iS})$
- Aggregate endowment $e = \sum_{i=1}^{I} e_i$; with slight abuse of notation, write $e = (e_0, e_1, \dots, e_S)$
- Agent i has HARA von Neumann-Morgenstern utility

$$u_i(x) = \frac{1}{a-1}(ax + b_i)^{1-1/a},$$

so a is common across agents but b_i arbitrary
- Agent i' utility function is

$$U_i(x) = u_i(x_0) + \beta \sum_{s=1}^{S} \pi_s u_i(x_s),$$

where discount factor $\beta > 0$ is common

State prices with HARA utility

Theorem

Let $\mathcal{E} = \{I, (e_i), (U_i)\}$ be an Arrow-Debreu economy with two periods, denoted by $t = 0, 1$, and S states at $t = 1$ (state s occurs with probability $\pi_s > 0$). Suppose that agent i has utility function

$$U_i(x_0, \dots, x_S) = u_i(x_0) + \beta \operatorname{E}[u_i(x_s)] = u_i(x_0) + \beta \sum_{s=1}^{S} \pi_s u_i(x_s),$$

where $\beta > 0$ is the (common) discount factor and $u_i(x)$ is a HARA Bernoulli utility function with parameters (a, b_i) (so a is common across agents). Normalizing the price of $t = 0$ good as $p_0 = 1$, the state price is then

$$p_s = \beta \pi_s \left(\frac{a e_s + b}{a e_0 + b} \right)^{-1/a},$$

where $b = \sum_{i=1}^{I} b_i$.

Proof

- By same argument as previous aggregation result, economy equivalent to one with a HARA representative agent with parameters (a, b), where $b = \sum_{i=1}^{I} b_i$
- Representative agent consumes aggregate endowment in equilibrium
- Let
$$L = u(x_0) + \beta \sum_{s=1}^{S} \pi_s u(x_s) + \lambda \left(p \cdot e - p \cdot x\right)$$
be Lagrangian
- Since $u'(x) = (ax + b)^{-1/a}$, first-order conditions are
$$\begin{aligned} \lambda p_0 &= u'(x_0) = u'(e_0) = (ae_0 + b)^{-1/a}, \\ \lambda p_s &= \beta \pi_s u'(x_s) = \beta \pi_s u'(e_s) = \beta \pi_s (ae_s + b)^{-1/a} \end{aligned}$$
- Dividing two equations and using $p_0 = 1$ yield desired result

Mutual Fund Theorem

Corollary (Mutual Fund Theorem)

Let everything be as above. Then any agent's consumption at $t = 1$ can be replicated just by the aggregate endowment ("stock market") and the vector of ones $1 = (1, \dots, 1)'$ ("risk-free asset").

Mutual Fund Theorem

Corollary (Mutual Fund Theorem)

Let everything be as above. Then any agent's consumption at $t = 1$ can be replicated just by the aggregate endowment ("stock market") and the vector of ones $\mathbf{1} = (1, \dots, 1)'$ ("risk-free asset").

- Mutual Fund Theorem has an enormous practical implication
- No matter what your risk attitude or traded assets are, the optimal portfolio is a combination of the aggregate stock market and the risk-free asset
- Thus all you need to decide is how much to invest in each asset
- Influenced by this theorem, John Bogle founded the Vanguard Group in 1974 and started offering the first index fund in 1975 (https://en.wikipedia.org/wiki/Index_fund)

Proof of Mutual Fund Theorem

- Let $\lambda_i > 0$ be Lagrange multiplier of agent i
- By first-order condition we obtain

$$\beta\pi_s(ax_s + b_i)^{-1/a} = \lambda_i p_s = \lambda_i \beta\pi_s \left(\frac{ae_s + b}{ae_0 + b}\right)^{-1/a}$$

$$\iff ax_s + b_i = \lambda_i^{-a} \frac{ae_s + b}{ae_0 + b}$$

$$\iff x_s = \lambda_i^{-a} \frac{1}{ae_0 + b} e_s + \frac{1}{a}\left(\lambda_i^{-a} \frac{b}{ae_0 + b} - b_i\right)$$

- Therefore there exist constants $\theta_i > 0$, $\phi_i \in \mathbb{R}$ such that $x_i = \theta_i e + \phi_i 1$, where $x_i = (x_{i1}, \dots, x_{iS})'$ and $e = (e_1, \dots, e_S)'$
- Hence agent i's consumption at $t = 1$ can be spanned by stock and risk-free asset

Stochastic discount factor

- Let us go back to general no-arbitrage pricing formula $q = \sum_{s=1}^{S} p_s A_s$, where
  - q: asset price,
  - p_s: state price,
  - A_s: asset payoff
- Can rewrite

$$q = \sum_{s=1}^{S} p_s A_s = \sum_{s=1}^{S} \pi_s \frac{p_s}{\pi_s} A_s = \mathrm{E}[mA],$$

  - π_s: probability of state s,
  - $m = (m_1, \dots, m_S)$ with $m_s = p_s/\pi_s$: stochastic discount factor (SDF)
- Modeling SDF is popular topic in theoretical and empirical finance (with HARA, $m_s = p_s/\pi_s = \beta \left(\frac{ae_s+b}{ae_0+b}\right)^{-1/a}$)

Covariance pricing

- Using SDF, asset pricing formula is

$$\text{Price} = \mathrm{E}[\text{SDF} \times \text{Payoff}]$$

- Dividing both sides by price, get

$$1 = \mathrm{E}\left[\text{SDF} \times \frac{\text{Payoff}}{\text{Price}}\right] = \mathrm{E}[\text{SDF} \times \text{Return}] = \mathrm{E}[mR]$$

- Letting R_f be gross risk-free rate, get

$$1 = \mathrm{E}[mR_f] = \mathrm{E}[m]R_f \iff R_f = 1/\,\mathrm{E}[m]$$

Covariance pricing

- Recall definition of covariance

$$\mathrm{Cov}[X, Y] = \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]$$

- Setting $X = m$ and $Y = R$, get

$$\begin{aligned} &\mathrm{Cov}[m, R] = \mathrm{E}[mR] - \mathrm{E}[m]\,\mathrm{E}[R] = 1 - \frac{\mathrm{E}[R]}{R_f} \\ \iff &\underbrace{\mathrm{E}[R] - R_f}_{\text{risk premium}} = -R_f \underbrace{\mathrm{Cov}[m, R]}_{\text{covariance}} \end{aligned}$$

- Risk premium of asset is (negatively) proportional to covariance between SDF and asset return (covariance pricing formula)

Capital Asset Pricing Model (CAPM)

- We now specialize covariance pricing formula to SDF for HARA utility, in particular quadratic utility
- Set $a = -1$; then HARA utility is

$$u(x) = \frac{1}{a-1}(ax+b)^{1-1/a} = -\frac{1}{2}(b-x)^2,$$

which is quadratic

- Recall that for HARA utility (with $a = -1$), SDF is

$$m_s = \frac{p_s}{\pi_s} = \beta \left(\frac{ae_s + b}{ae_0 + b} \right)^{-1/a} = \beta \frac{b - e_s}{b - e_0}$$

- Clearly m is affine in endowment e, and hence also affine in market return $R_m = e/P$ (think of market return as claim to aggregate endowment, which is similar to index fund)
- Hence can write $m = A - BR_m$ for some constants A, B

Capital Asset Pricing Model (CAPM)

- Plug in SDF $m = A - BR_m$ into covariance pricing formula

$$\mathrm{E}[R_j] - R_f = -R_f \operatorname{Cov}[m, R_j],$$

  where R_j is gross return on asset (or portfolio) j
- Then

$$\begin{aligned} \mathrm{E}[R_j] - R_f &= -R_f \operatorname{Cov}[m, R_j] \\ &= -R_f \operatorname{Cov}[A - BR_m, R_j] = BR_f \operatorname{Cov}[R_m, R_j] \end{aligned}$$

- Specializing to $R_j = R_m$ (market return), get

$$\mathrm{E}[R_m] - R_f = BR_f \operatorname{Cov}[R_m, R_m] = BR_f \operatorname{Var}[R_m]$$

Capital Asset Pricing Model (CAPM)

- We know

$$
\begin{aligned}
\mathrm{E}[R_j] - R_f &= BR_f \operatorname{Cov}[R_m, R_j] \quad \text{for any asset } j, \\
\mathrm{E}[R_m] - R_f &= BR_f \operatorname{Var}[R_m]
\end{aligned}
$$

- Dividing first equation by second to eliminate unknown constant B, get

$$
\begin{aligned}
\underbrace{\mathrm{E}[R_j] - R_f}_{\text{asset } j\text{'s risk premium}} &= \frac{\operatorname{Cov}[R_m, R_j]}{\operatorname{Var}[R_m]}(\mathrm{E}[R_m] - R_f) \\
&= \beta_j \underbrace{(\mathrm{E}[R_m] - R_f)}_{\text{market risk premium}},
\end{aligned}
$$

where $\beta_j := \operatorname{Cov}[R_m, R_j] / \operatorname{Var}[R_m]$ is called asset j's beta

Implication of CAPM

- CAPM predicts that an asset's risk premium is proportional to market risk premium
- Constant of proportionality is beta, which is covariance of asset and market return (scaled by variance of market return)
- Can estimate β_j by running time series regression

$$R_{jt} - R_{ft} = \alpha_j + \beta_j (R_{mt} - R_{ft}) + \epsilon_{jt}$$

- Theory predicts $\alpha_j = 0$; assets with $\alpha_j > 0$ (< 0) have abnormally high (low) returns, and hence are undervalued (overvalued)
- Can be used for portfolio management

Security market line (SML)

- Linear relationship between beta and risk premium is called security market line