Economics 205 Final Examination

Professors Toda and Watson

Fall 2015

Name: ______________________________

Instructions:

- You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets. Calculators are allowed.
- All logarithms are base $\mathrm{e} = 2.718281828\ldots$, so $\ln x$ and $\log x$ are the same.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

1. (10 points) Consider the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = \frac{2n-1}{n+1}$ for every $n \in \mathbb{N}$. Find the limit of this sequence. Provide proof of this limit by describing, for a given $\varepsilon > 0$, a positive integer N that satisfies the definition of the limit.

2. (10 points) Calculate the following:

   (a) $\lim\limits_{x \to 0} x \ln x^2$

   (b) $\displaystyle\int_e^{e^3} (4x \ln x) dx$

3. (10 points) Consider the function $f\colon [1, \infty) \to \mathbb{R}$ defined by $f(x) \equiv 1/x$.

   (a) Calculate the first four derivatives of f and write a general expression for $f^{(k)}$ (the kth derivative of f).

(b) Write the third degree Taylor polynomial of f, centered at the point $c = 1$, as a function of x.

(c) Suppose you want to use a Taylor polynomial to estimate $f(x)$ for values of x in the interval $[1, 3]$. Write the error term of the nth degree Taylor polynomial, as a function of x, n, and t. Center the polynomial around $c = 1$. Can you find a value of n such that the absolute value of the error term guaranteed to be less than $1/10$ for all $x \in [1, 3]$, without knowing the value of t between c and x? If so, provide such a number.

4. (10 points) Suppose X and Y are open intervals of $\mathbb{R}$. Consider two functions, $g\colon X \to Y$ and $f\colon Y \to \mathbb{R}$. Evaluate the following claim:

   Claim: If $\lim\limits_{x\to a} g(x)$ exists and equals $b \in Y$, and if $\lim\limits_{y\to b} f(y)$ exists and equals $c \in \mathbb{R}$, then $\lim\limits_{x\to a}(f \circ g)(x)$ exists and equals c.

   Is this claim correct? If you answer "yes," explain how you would prove it. Be as formal as you can. If you answer "no," provide a counterexample.

5. (10 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be arbitrary functions.

   (a) What assumptions are needed to guarantee that $\inf\{|f(x) - g(x)| \mid x \in \mathbb{R}\}$ exists?

   (b) What assumptions are needed to guarantee that $\min\{|f(x) - g(x)| \mid x \in \mathbb{R}\}$ exists?

6. (10 points) Let $f(x) = \sqrt{x^2 + 1}$. Compute $f'(x)$, $f''(x)$, and determine whether f is convex, concave, or neither.

7. Let $f(x_1, x_2) = x_1^3 + 3x_1^2 + x_1x_2 + x_2^2 - 5x_2 + 6$.

   (a) (2 points) Compute the gradient and the Hessian of f.

(b) (4 points) Find the stationary point(s) of f.

(c) (4 points) Determine whether each stationary point is a local maximum, local minimum, or a saddle point.

8. Consider the problem

$$\begin{array}{ll} \text{minimize} & 3x_1 + x_2 \\ \text{subject to} & x_2 \leq -x_1^2, \\ & x_1^2 + (x_2 - 1)^2 \leq 1. \end{array}$$

(a) (2 points) Draw a picture of the set that each constraint defines (in one picture).

(b) (2 points) Compute the solution.

(c) (3 points) Compute the tangent cone and the linearizing cone at the solution.

(d) (3 points) Do the Karush-Kuhn-Tucker conditions hold? If so, give the Lagrange multipliers. If not, explain why.

9. Consider the problem

$$\begin{array}{ll} \text{maximize} & \log x_1 + u_1 \log x_2 \\ \text{subject to} & x_1 + x_2 \leq u_2, \end{array}$$

where $x_1, x_2 > 0$ are variables and $u_1, u_2 > 0$ are parameters.

(a) (2 points) Prove that the objective function is concave. (Hint: by definition f is concave if $-f$ is convex.)

(b) (1 point) Write down the Lagrangian.

(c) (3 points) Compute the solution.

(d) (4 points) Let $\phi(u_1, u_2)$ be the maximum value of the problem. Compute $\frac{\partial \phi}{\partial u_1}$ and $\frac{\partial \phi}{\partial u_2}$.

10. (10 points) What is the separating hyperplane theorem? State the assumptions as well as the conclusion of both for weak separation and strict separation.