Economics 205 Final Examination

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Name: ______________________________

Instructions:

- You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets. Calculators are allowed.
- All logarithms are base $e = 2.718281828\ldots$, so $\ln x$ and $\log x$ are the same.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

1. (10 points) Consider the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = \frac{2n-1}{n+1}$ for every $n \in \mathbb{N}$. Find the limit of this sequence. Provide proof of this limit by describing, for a given $\varepsilon > 0$, a positive integer N that satisfies the definition of the limit.

Solution: 1. The limit clearly equals 2. Note that $\left|\frac{2n-1}{n+1} - 2\right| = \frac{3}{n+1}$. Therefore $\left|\frac{2n-1}{n+1} - 1\right| < \varepsilon$ is equivalent to $\frac{3}{n+1} < \varepsilon$, which simplifies to $\frac{3}{\varepsilon} - 1 < n$. Thus, for any given $\varepsilon > 0$, let N be the smallest integer satisfying this inequality.

2. (10 points) Calculate the following:
   (a) $\lim_{x \to 0} x \ln x^2$

   Solution: 0

   (b) $\int_e^{e^3} (4x \ln x) dx$

   Solution: Integration by parts: $5e^6 - e^2$.

3. (10 points) Consider the function $f \colon [1, \infty) \to \mathbb{R}$ defined by $f(x) \equiv 1/x$.
   (a) Calculate the first four derivatives of f and write a general expression for $f^{(k)}$ (the kth derivative of f).

   Solution: $f^{(k)}(x) = (-1)^k k! x^{-(k+1)}$.

   (b) Write the third degree Taylor polynomial of f, centered at the point $c = 1$, as a function of x.

   Solution: $P_3(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3$, which equals

   $$P_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3.$$

   (c) Suppose you want to use a Taylor polynomial to estimate $f(x)$ for values of x in the interval $[1, 3]$. Write the error term of the nth degree Taylor polynomial, as a function of x, n, and t. Center the polynomial around $c = 1$. Can you find a value of n such that the absolute value of the error term guaranteed to be less than $1/10$ for all $x \in [1, 3]$, without knowing the value of t between c and x? If so, provide such a number.

   Solution: The error term of the Taylor polynomial centered at $c = 1$ is

   $$E_n(x) = \frac{(-1)^{n+1}(n+1)!(x-1)^{n+1}}{(n+1)!t^{n+2}},$$

so we have

$$|E_n(x)| = \frac{(x-1)^{n+1}}{t^{n+2}},$$

which cannot be bounded by $1/10$ for $1 \leq t \leq x$ for any n.

4. (10 points) Suppose X and Y are open intervals of $\mathbb{R}$. Consider two functions, $g\colon X \to Y$ and $f\colon Y \to \mathbb{R}$. Evaluate the following claim:

   Claim: If $\lim\limits_{x\to a} g(x)$ exists and equals $b \in Y$, and if $\lim\limits_{y\to b} f(y)$ exists and equals $c \in \mathbb{R}$, then $\lim\limits_{x\to a}(f \circ g)(x)$ exists and equals c.

   Is this claim correct? If you answer "yes," explain how you would prove it. Be as formal as you can. If you answer "no," provide a counterexample.

   Solution: No. A counterexample is given by $g(x) = 0$ for all x, and $f(y) = 0$ if $y \neq 0$ and $f(0) = 1$. Let $a = b = 0$. Note that $\lim_{x\to 0} g(x) = 0$ and $\lim_{y\to 0} f(x) = 0$, but $\lim_{x\to 0}(f \circ g)(x) = 1$.

5. (10 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be arbitrary functions.

   (a) What assumptions are needed to guarantee that $\inf\{|f(x) - g(x)| \mid x \in \mathbb{R}\}$ exists?

   Solution: No assumptions are needed.

   (b) What assumptions are needed to guarantee that $\min\{|f(x) - g(x)| \mid x \in \mathbb{R}\}$ exists?

   Solution: Continuity of the two functions is sufficient.

6. (10 points) Let $f(x) = \sqrt{x^2+1}$. Compute $f'(x)$, $f''(x)$, and determine whether f is convex, concave, or neither.

   Solution: $f'(x) = \frac{x}{\sqrt{x^2+1}}$, $f''(x) = (x^2+1)^{-\frac{3}{2}} > 0$, so f is convex.

7. Let $f(x_1, x_2) = x_1^3 + 3x_1^2 + x_1x_2 + x_2^2 - 5x_2 + 6$.

   (a) (2 points) Compute the gradient and the Hessian of f.

Solution:

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 + 6x_1 + x_2 \\ x_1 + 2x_2 - 5 \end{bmatrix},$$
$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 + 6 & 1 \\ 1 & 2 \end{bmatrix}.$$

(2 points for the gradient, 1 point for the Hessian.)

(b) (4 points) Find the stationary point(s) of f.

Solution: By the first-order condition, we get

$$\nabla f(x_1, x_2) = 0 \iff \begin{bmatrix} 3x_1^2 + 6x_1 + x_2 \\ x_1 + 2x_2 - 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From the first equation, we get $x_2 = -3x_1^2 - 6x_1$. Substituting into the second equation, we get

$$\begin{aligned} x_1 + 2(-3x_1^2 - 6x_1) - 5 = 0 &\iff 6x_1^2 + 11x_1 + 5 = 0 \\ &\iff (6x_1 + 5)(x_1 + 1) = 0 \\ &\iff x_1 = -1, -\frac{5}{6}. \end{aligned}$$

If $x_1 = -1$, then $x_2 = 3$. If $x_1 = -\frac{5}{6}$, then $x_2 = \frac{35}{12}$. Therefore the stationary points are

$$(x_1, x_2) = (-1, 3), \left(-\frac{5}{6}, \frac{35}{12}\right).$$

(c) (4 points) Determine whether each stationary point is a local maximum, local minimum, or a saddle point.

Solution: At $x_1 = -1$, the Hessian is

$$H = \nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Since the determinant is $0 \times 2 - 1^2 = -1 < 0$, it is a saddle point.
At $x_1 = -\frac{5}{6}$, the Hessian is

$$H = \nabla^2 f(x_1, x_2) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Since $1 > 0$ and $1 \times 2 - 1^2 = 1 > 0$, it is positive definite. Therefore it is a local minimum.

8. Consider the problem

$$\begin{array}{ll} \text{minimize} & 3x_1 + x_2 \\ \text{subject to} & x_2 \le -x_1^2, \\ & x_1^2 + (x_2-1)^2 \le 1. \end{array}$$

(a) (2 points) Draw a picture of the set that each constraint defines (in one picture).

> **Solution:** The constraint $x_2 \le -x_1^2$ is a parabola with apex $(0,0)$ (1 point). The constraint $x_1^2 + (x_2-1)^2 \le 1$ is a disk with center $(0,1)$ and radius 1 (1 point).

(b) (2 points) Compute the solution.

> **Solution:** Since the constraint consists of the single point $(x_1, x_2) = (0,0)$, it is the unique solution.

(c) (3 points) Compute the tangent cone and the linearizing cone at the solution.

> **Solution:** Let $\bar{x} = (x_1, x_2) = (0,0)$. Since the constraint set is $C = \{\bar{x}\}$, we have $\bar{x} + ty \in C$ with $t > 0$ only if $y = (0,0)$. Therefore the tangent cone is $T(\bar{x}) = \{0\}$ (1 point).
> Let $g_1(x) = x_1^2 + x_2$ and $g_2(x) = x_1^2 + (x_2-1)^2 - 1$. Then
>
> $$\nabla g_1(\bar{x}) = \begin{bmatrix} 2x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla g_2(\bar{x}) = \begin{bmatrix} 2x_1 \\ 2(x_2-1) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix},$$
>
> so the linearizing cone is $L(\bar{x}) = \{(y_1, y_2) \mid y_2 = 0\}$ (2 points).

(d) (3 points) Do the Karush-Kuhn-Tucker conditions hold? If so, give the Lagrange multipliers. If not, explain why.

> **Solution:** Let
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> $$L(x_1, x_2, \lambda_1, \lambda_2) = 3x_1 + x_2 + \lambda_1(x_1^2 + x_2) + \lambda_2(x_1^2 + (x_2-1)^2 - 1)$$
>
> be the Lagrangian. If the KKT theorem holds, we must have
>
> $$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} = 3 + 2\lambda_1 x_1 + 2\lambda_2 x_1, \\ 0 &= \frac{\partial L}{\partial x_2} = 1 + \lambda_1 + 2\lambda_2(x_2 - 1) \end{aligned}$$
>
> at $\bar{x} = (x_1, x_2) = (0,0)$, but the first equation becomes $0 = 3$, a contradiction. Therefore the KKT conditions do not hold (1 point).

The reason is because the Guignard constraint qualification $L(\bar{x}) \subset \operatorname{co} T(\bar{x})$ does not hold (2 points).

9. Consider the problem

$$\begin{array}{ll} \text{maximize} & \log x_1 + u_1 \log x_2 \\ \text{subject to} & x_1 + x_2 \leq u_2, \end{array}$$

where $x_1, x_2 > 0$ are variables and $u_1, u_2 > 0$ are parameters.

(a) (2 points) Prove that the objective function is concave. (Hint: by definition f is concave if $-f$ is convex.)

Solution: Since $(\log x)'' = (1/x)' = -1/x^2 < 0$, $f(x) = \log x$ is concave. Since the objective function is $f(x_1) + u_1 f(x_2)$, it is concave.

(b) (1 point) Write down the Lagrangian.

Solution:

$$L(x_1, x_2, \lambda, u_1, u_2) = \log x_1 + u_1 \log x_2 + \lambda(u_2 - x_1 - x_2).$$

(c) (3 points) Compute the solution.

Solution: Since the objective function is concave, the constraint is convex, and the Slater condition holds, the KKT conditions are necessary and sufficient for a solution. The first-order condition is

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} = \frac{1}{x_1} - \lambda \iff x_1 = \frac{1}{\lambda}, \\ 0 &= \frac{\partial L}{\partial x_2} = \frac{u_1}{x_2} - \lambda \iff x_2 = \frac{u_1}{\lambda}. \end{aligned}$$

Clearly $\lambda > 0$. By complementary slackness, we get $x_1 + x_2 = u_2$, so $\lambda = \frac{1+u_1}{u_2}$ and the solution is

$$(x_1, x_2) = \left(\frac{u_2}{1+u_1}, \frac{u_1 u_2}{1+u_1} \right).$$

(d) (4 points) Let $\phi(u_1, u_2)$ be the maximum value of the problem. Compute $\frac{\partial \phi}{\partial u_1}$ and $\frac{\partial \phi}{\partial u_2}$.

Solution: Let $u = (u_1, u_2)$. By the envelope theorem,

$$\begin{bmatrix} \frac{\partial \phi}{\partial u_1} \\ \frac{\partial \phi}{\partial u_2} \end{bmatrix} = \nabla_u \phi(u) = \nabla_u L(x(u), \lambda(u), u)$$
$$= \begin{bmatrix} \log x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} \log \frac{u_1 u_2}{1+u_1} \\ \frac{1+u_1}{u_2} \end{bmatrix}.$$

Alternatively, you can compute $\phi(u_1, u_2)$ and its partial derivatives.

10. (10 points) What is the separating hyperplane theorem? State the assumptions as well as the conclusion of both for weak separation and strict separation.

Solution: Let C, D be nonempty and convex. If $C \cap D = \emptyset$, then there exists $a \neq 0$ such that

$$\sup_{x \in C} \langle a, x \rangle \leq \inf_{x \in D} \langle a, x \rangle .$$

If in addition C is closed and D is compact, a can be taken such that

$$\sup_{x \in C} \langle a, x \rangle < \inf_{x \in D} \langle a, x \rangle .$$