Economics 205 Final Examination

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Name: ______________________________

Instructions:

- You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets. Calculators are allowed.
- All logarithms are base e $= 2.718281828\ldots$, so $\ln x$ and $\log x$ are the same.

Question:	1	2	3	4	5	6	Total
Points:	0	0	0	0	0	0	0
Score:							

1. Let K and K' be two compact sets in $\mathbb{R}$. We will define their *sum* $K + K' \subseteq \mathbb{R}$ as follows:
$$K + K' = \{z \in \mathbb{R} \mid z = x + y, (x, y) \in K \times K'\}.$$
   (a) Give the characterization of compactness in terms of sequences.

   (b) Use (a) to show that $K + K'$ is compact.

2. *This question consists of two parts which are largely independent. If you cannot establish the results required in (a), you can proceed with (b) by assuming (a) has been shown.*

   Let (a, b) be a couple of strictly positive real numbers (i.e. $a > 0$, $b > 0$), and let r be real (i.e. $r \in \mathbb{R}$). Define
$$M_r(a, b) = \left[\frac{a^r + b^r}{2}\right]^{1/r}, \quad \text{if } r \neq 0,$$
   and
$$M_0(a, b) = \sqrt{ab}.$$
   (a) Show that for any $a > 0$ and $b > 0$, $M_r(a, b)$ is continuous (as a function of r) at zero, i.e. that for any $a > 0$ and $b > 0$,
$$\lim_{r \to 0} M_r(a, b) = M_0(a, b).$$
   (Hint: use the fact that for any $x > 0$ and $y \in \mathbb{R}$, we can write $x^y = \exp(y \ln x)$.)

(b) We focus on $M_{-1}(a,b)$, $M_0(a,b)$, and $M_1(a,b)$, which are called, respectively, the harmonic mean, geometric mean, and ordinary arithmetic mean of a and b. Show that we have

$$M_0(a,b) \leq M_1(a,b).$$

(Hint: combine the square of a sum with the square of a difference.) Show that in addition

$$M_{-1}(a,b) \leq M_0(a,b).$$

(Hint: write $M_{-1}(a,b)$ as a function of $M_1(1/a, 1/b)$.)

3. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ and let $\mathbf{A} = \mathbf{x}\mathbf{x}^\top$. What is the rank of $\mathbf{A}$? (hint: consider the following two cases: (1) $\mathbf{x} = \mathbf{0}$ and (2) $\mathbf{x} \neq \mathbf{0}$, and compute the rank of $\mathbf{A}$ in each case.)

4. Consider an agent with utility function

$$U(x_1, x_2) = (1 - \alpha) \log x_1 + \alpha \log x_2,$$

where $0 < \alpha < 1$. Suppose that the prices of goods 1, 2 are p_1, p_2. Consider the expenditure minimization problem

$$\begin{array}{ll} \text{minimize} & p_1 x_1 + p_2 x_2 \\ \text{subject to} & U(x_1, x_2) \geq u, \\ & x_1, x_2 \geq 0, \end{array}$$

where $u \in \mathbb{R}$ is the target utility level.

(a) Are the Karush-Kuhn-Tucker conditions necessary for a solution? Answer yes or no, then explain why.

(b) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? Answer yes or no, then explain why.

(c) Let $e(p_1, p_2, u)$ be the minimum expenditure and $x_l(p_1, p_2, u)$ be the optimal demand, where $l = 1, 2$. Compute x_1, x_2.

(d) Compute $\partial e(p_1, p_2, u)/\partial p_1$.

5. Consider an agent who can invest in two assets, a stock and a risk-free bond. Let $R_f > 0$ be the gross risk-free rate and R be the gross return of the stock, which can take S different values $R_1, \dots, R_S > 0$ with probability $\pi_1, \dots, \pi_S$. Suppose that the agent wants to maximize the log portfolio return

$$v(\theta) := \mathrm{E}[\log(R\theta + R_f(1-\theta))] = \sum_{s=1}^{S} \pi_s \log(R_s\theta + R_f(1-\theta)),$$

where θ is the fraction of wealth invested in the stock. (It is allowed to shortsell the stock ($\theta < 0$) as well as buy the stock on margin ($\theta > 1$).) Let θ^* be the optimal portfolio.

(a) Show that the objective function v is strictly concave.

(b) Show that θ^* is unique.

(c) Derive the first-order condition.

(d) Show that $\partial\theta^*/\partial R_f < 0$, so if the risk-free rate goes up, the agent invests less in stock.

6. (a) What is the definition of a convex function?

(b) What is the definition of a quasi-convex function?

(c) Let $g_1, g_2 : \mathbb{R}^N \to \mathbb{R}$ be convex, and $h : \mathbb{R}^2 \to \mathbb{R}$ be increasing and quasi-convex. (h is increasing if $h(x_1, x_2) \leq h(y_1, y_2)$ whenever $x_1 \leq y_1$ and $x_2 \leq y_2$.) Define $f : \mathbb{R}^N \to \mathbb{R}$ by $f(x) = h(g_1(x), g_2(x))$. Prove that h is quasi-convex.