Economics 205 Final Examination

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Name: ____

Instructions:

- You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets. Calculators are allowed.
- All logarithms are base $e = 2.718281828\ldots$, so $\ln x$ and $\log x$ are the same.

Question:	1	2	3	4	5	6	Total
Points:	0	0	0	0	0	0	0
Score:							

1. Let K and K' be two compact sets in $\mathbb{R}$. We will define their *sum* $K + K' \subseteq \mathbb{R}$ as follows:

$$K + K' = \{z \in \mathbb{R} \mid z = x + y, (x, y) \in K \times K'\}.$$

   (a) Give the characterization of compactness in terms of sequences.

   Solution:

   (b) Use (a) to show that $K + K'$ is compact.

   Solution:

2. *This question consists of two parts which are largely independent. If you cannot establish the results required in (a), you can proceed with (b) by assuming (a) has been shown.*

   Let (a, b) be a couple of strictly positive real numbers (i.e. $a > 0$, $b > 0$), and let r be real (i.e. $r \in \mathbb{R}$). Define

$$M_r(a, b) = \left[\frac{a^r + b^r}{2}\right]^{1/r}, \quad \text{if } r \neq 0,$$

   and

$$M_0(a, b) = \sqrt{ab}.$$

   (a) Show that for any $a > 0$ and $b > 0$, $M_r(a, b)$ is continuous (as a function of r) at zero, i.e. that for any $a > 0$ and $b > 0$,

$$\lim_{r \to 0} M_r(a, b) = M_0(a, b).$$

   (Hint: use the fact that for any $x > 0$ and $y \in \mathbb{R}$, we can write $x^y = \exp(y \ln x)$.)

   Solution:

   (b) We focus on $M_{-1}(a, b)$, $M_0(a, b)$, and $M_1(a, b)$, which are called, respectively, the harmonic mean, geometric mean, and ordinary arithmetic mean of a and b. Show that we have

$$M_0(a, b) \leq M_1(a, b).$$

   (Hint: combine the square of a sum with the square of a difference.) Show that in addition

$$M_{-1}(a, b) \leq M_0(a, b).$$

   (Hint: write $M_{-1}(a, b)$ as a function of $M_1(1/a, 1/b)$.)

   Solution:

3. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ and let $\mathbf{A} = \mathbf{x}\mathbf{x}^\top$. What is the rank of $\mathbf{A}$? (hint: consider the following two cases: (1) $\mathbf{x} = \mathbf{0}$ and (2) $\mathbf{x} \neq \mathbf{0}$, and compute the rank of $\mathbf{A}$ in each case.)

Solution:

4. Consider an agent with utility function

$$U(x_1, x_2) = (1 - \alpha) \log x_1 + \alpha \log x_2,$$

where $0 < \alpha < 1$. Suppose that the prices of goods 1, 2 are p_1, p_2. Consider the expenditure minimization problem

$$\begin{aligned} &\text{minimize} && p_1x_1 + p_2x_2 \\ &\text{subject to} && U(x_1, x_2) \geq u, \\ &&& x_1, x_2 \geq 0, \end{aligned}$$

where $u \in \mathbb{R}$ is the target utility level.

(a) Are the Karush-Kuhn-Tucker conditions necessary for a solution? Answer yes or no, then explain why.

Solution: Yes. The problem is equivalent to

$$\begin{aligned} &\text{minimize} && p_1x_1 + p_2x_2 \\ &\text{subject to} && -(1 - \alpha) \log x_1 - \alpha \log x_2 + u \leq 0, \\ &&& -x_1 \leq 0, \\ &&& -x_2 \leq 0. \end{aligned}$$

Clearly the objective function is linear (hence convex). The constraint functions are all convex since $\log x$ is concave. Therefore the problem is a convex minimization problem. Furthermore, the Slater constraint qualification holds by taking x_1, x_2 large enough. Hence the assumptions of the KKT theorem are satisfied.

(b) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? Answer yes or no, then explain why.

Solution: Since the problem is a convex minimization problem, the KKT conditions are sufficient for optimality.

(c) Let $e(p_1, p_2, u)$ be the minimum expenditure and $x_l(p_1, p_2, u)$ be the optimal demand, where $l = 1, 2$. Compute x_1, x_2.

Solution: Clearly the constraints $x_l \geq 0$ do not bind, since utility is $-\infty$ when $x_l = 0$. Hence we can ignore these constraints. The Lagrangian is

$$L(x, \lambda) = p_1x_1 + p_2x_2 + \lambda(-(1 - \alpha) \log x_1 - \alpha \log x_2 + u).$$

The first-order conditions are $p_1 - \lambda(1-\alpha)/x_1 = 0$ and $p_2 - \lambda\alpha/x_2 = 0$, so $x_1 = \frac{\lambda(1-\alpha)}{p_1}$ and $x_2 = \frac{\lambda\alpha}{p_2}$. Clearly $\lambda > 0$, so by complementary slackness, we have

$$u = (1-\alpha)\log x_1 + \alpha \log x_2 = \log \lambda - (1-\alpha)\log p_1 - \alpha \log p_2 + H(\alpha),$$

where $H(\alpha) := \alpha \log \alpha + (1-\alpha)\log(1-\alpha)$. Taking the exponential, we get

$$\lambda = \mathrm{e}^u p_1^{1-\alpha} p_2^{\alpha} \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}.$$

Therefore

$$\begin{aligned} x_1 &= \mathrm{e}^u \left(\frac{p_2}{p_1}\frac{1-\alpha}{\alpha}\right)^{\alpha}, \\ x_2 &= \mathrm{e}^u \left(\frac{p_1}{p_2}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}. \end{aligned}$$

(d) Compute $\partial e(p_1, p_2, u)/\partial p_1$.

Solution: By the envelope theorem,

$$\frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{\partial L}{\partial p_1} = x_1.$$

5. Consider an agent who can invest in two assets, a stock and a risk-free bond. Let $R_f > 0$ be the gross risk-free rate and R be the gross return of the stock, which can take S different values $R_1, \dots, R_S > 0$ with probability $\pi_1, \dots, \pi_S$. Suppose that the agent wants to maximize the log portfolio return

$$v(\theta) := \mathrm{E}[\log(R\theta + R_f(1-\theta))] = \sum_{s=1}^{S} \pi_s \log(R_s\theta + R_f(1-\theta)),$$

where θ is the fraction of wealth invested in the stock. (It is allowed to shortsell the stock ($\theta < 0$) as well as buy the stock on margin ($\theta > 1$).) Let θ^* be the optimal portfolio.

(a) Show that the objective function v is strictly concave.

Solution:

$$v''(\theta) = -\sum_{s=1}^{S} \pi_s \frac{(R_s - R_f)^2}{(R_s\theta + R_f(1-\theta))^2} < 0,$$

so v is strictly concave.

(b) Show that θ^* is unique.

Solution: Suppose that $\theta_1 < \theta_2$ are two solutions. Then $v(\theta_1) = v(\theta_2)$. Take any $\alpha \in (0,1)$. By strict concavity, we have

$$v((1-\alpha)\theta_1 + \alpha\theta_2) > (1-\alpha)v(\theta_1) + \alpha v(\theta_2) = v(\theta_1),$$

so the portfolio $\theta_3 = (1-\alpha)\theta_1 + \alpha\theta_2$ gives higher utility than θ_1, which is a contradiction.

(c) Derive the first-order condition.

Solution:

$$v'(\theta) = \mathrm{E}\left[\frac{R - R_f}{R\theta + R_f(1-\theta)}\right] = 0.$$

(d) Show that $\partial\theta^*/\partial R_f < 0$, so if the risk-free rate goes up, the agent invests less in stock.

Solution: Let

$$F(R_f, \theta) = \mathrm{E}\left[\frac{R - R_f}{R\theta + R_f(1-\theta)}\right].$$

Then $F(R_f, \theta^*) = 0$. By the implicit function theorem, we have

$$\frac{\partial\theta^*}{\partial R_f} = -\frac{D_{R_f}F}{D_\theta F}.$$

By simple algebra,

$$\begin{aligned} D_\theta F &= \frac{\partial F}{\partial \theta} = -\mathrm{E}\left[\frac{(R-R_f)^2}{(R\theta + R_f(1-\theta))^2}\right] < 0, \\ D_{R_f}F &= \frac{\partial F}{\partial R_f} = \mathrm{E}\left[\frac{-(R\theta + R_f(1-\theta)) - (R-R_f)(1-\theta)}{(R\theta + R_f(1-\theta))^2}\right] \\ &= -\mathrm{E}\left[\frac{R}{(R\theta + R_f(1-\theta))^2}\right] < 0, \end{aligned}$$

so $\partial\theta^*/\partial R_f < 0$.

6. (a) What is the definition of a convex function?

Solution: Let Ω be a convex set. $f : \Omega \to \mathbb{R}$ is convex if for all $x_1, x_2 \in \Omega$ and $\alpha \in [0,1]$, we have $f((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)f(x_1) + \alpha f(x_2)$.

(b) What is the definition of a quasi-convex function?

Solution: Let Ω be a convex set. $f : \Omega \to \mathbb{R}$ is quasi-convex if for all $x_1, x_2 \in \Omega$ and $\alpha \in [0,1]$, we have $f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}$.

(c) Let $g_1, g_2 : \mathbb{R}^N \to \mathbb{R}$ be convex, and $h : \mathbb{R}^2 \to \mathbb{R}$ be increasing and quasi-convex. (h is increasing if $h(x_1, x_2) \leq h(y_1, y_2)$ whenever $x_1 \leq y_1$ and $x_2 \leq y_2$.) Define $f : \mathbb{R}^N \to \mathbb{R}$ by $f(x) = h(g_1(x), g_2(x))$. Prove that h is quasi-convex.

Solution: Take any $x_1, x_2 \in \mathbb{R}^N$ and $\alpha \in [0,1]$. Since g_i is convex, we have $g_i((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)g_i(x_1) + \alpha g_i(x_2)$. Define $G : \mathbb{R}^N \to \mathbb{R}^2$ by $G(x) = \left[\begin{smallmatrix} g_1(x) \\ g_2(x) \end{smallmatrix}\right]$. Then

$$G((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)G(x_1) + \alpha G(x_2),$$

where the inequality is component-wise. Applying h to both sides, and using the fact that h is increasing and quasi-convex, it follows that

$$\begin{aligned} f((1-\alpha)x_1 + \alpha x_2) &= h(G((1-\alpha)x_1 + \alpha x_2)) \\ &\leq h((1-\alpha)G(x_1) + \alpha G(x_2)) \\ &\leq \max\{h(G(x_1)), h(G(x_2))\} = \max\{f(x_1), f(x_2)\}, \end{aligned}$$

so f is quasi-convex.