Economics 205 Final Exam

Alexis Akira Toda

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Instructions:

- You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets. Calculators are not allowed.
- All logarithms are base $\mathrm{e} = 2.718281828\ldots$, so $\ln x$ and $\log x$ are the same.
- Questions are not necessarily ordered in the order of difficulty, and some parts are (far) easier than others. Make sure to look at all questions and parts.

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	120
Score:							

1. Let A be an $N \times N$ positive matrix, so $A = (a_{mn})$ with $a_{mn} > 0$ for all $1 \le m, n \le N$. Let $\rho(A)$ be the spectral radius (largest absolute value of all eigenvalues) of A. We know from the Perron-Frobenius theorem that $\alpha = \rho(A) > 0$ is an eigenvalue of A and there exist (unique up to normalization) positive right and left eigenvectors corresponding to α, so $Ax = \alpha x$ and $y'A = \alpha y'$. Normalize the eigenvectors such that they have (Euclidean) norm 1, so

$$\|x\| = \sqrt{\sum_{n=1}^{N} x_n^2} = 1.$$

This question asks you to prove that the Perron root α and Perron vector x are smooth functions of the elements of A. Define $F : \mathbb{R}_{++}^{N^2} \times \mathbb{R}_{++}^{N} \times \mathbb{R}_{++} \to \mathbb{R}^N \times \mathbb{R}$ by

$$F(A, x, \alpha) = \begin{bmatrix} (A - \alpha I)x \\ \|x\|^2 - 1 \end{bmatrix}.$$

(a) (5 points) Compute the Jacobian $J = D_{(x,\alpha)}F$ of F with respect to (x, α).

(b) (10 points) Prove that J is regular at (A, x, α). (Hint: Assume $J \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for some $u \in \mathbb{R}^N$, $v \in \mathbb{R}$, and show that $u = 0$ and $v = 0$.)

(c) (5 points) Prove that the Perron root α and Perron vector x are continuously differentiable in the elements of A.

2. Let

$$A = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix},$$

where $0 < p, q < 1$.

(a) (10 points) Compute all eigenvalues of A.

(b) (10 points) Compute the right and left eigenvectors of A corresponding to the eigenvalue with largest absolute value. Normalize the eigenvalues such that the sum of absolute values of elements (L^1 norm) is 1.

3. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad (i = 1, \dots, I) \end{array}$$

where $f, g_i : \mathbb{R}^N \to \mathbb{R}$ are differentiable.

(a) (4 points) What is the definition of a convex function?

(b) (4 points) What is the definition of a quasi-convex function?

(c) (4 points) Suppose that g_i's are convex. What is the Slater condition (constraint qualification)?

(d) (8 points) The Karush-Kuhn-Tucker theorem (for convex functions) says that if g_i's are convex and satisfy the Slater condition, then a solution to the optimization problem satisfies the first-order and complementary slackness conditions. Is this also true if g_i's are only quasi-convex? If so, prove it. If not, provide a counterexample.

4. Consider the utility maximization problem

$$\begin{aligned} &\text{maximize} && u(x_1, x_2) = \alpha \log x_1 + (1-\alpha) \log x_2 \\ &\text{subject to} && p_1 x_1 + p_2 x_2 \le w, \end{aligned}$$

where $\alpha \in (0,1)$ is a preference parameter, $x_1, x_2 > 0$ are consumption of goods 1, 2, p_1, p_2 are prices, and $w > 0$ is wealth.

(a) (10 points) Solve the utility maximization problem. Make sure that all of your arguments are rigorous.

(b) (10 points) Let $V(p_1, p_2, w, \alpha)$ be the maximized utility as a function of all parameters. Derive a condition on these parameters such that liking good 1 more makes you happier, i.e., $\frac{\partial}{\partial \alpha} V > 0$.

5. Consider an infinite-horizon optimal consumption-saving problem with stochastic returns. Suppose the agent has utility function

$$\mathrm{E} \sum_{t=0}^{\infty} \beta^t \log c_t,$$

where E denotes the expectation, $\beta \in (0,1)$ is the discount factor, and $c_t > 0$ is consumption at time t. Suppose that there are S states indexed by $s = 1, \dots, S$. The state at time t, denoted by s_t, evolves according to a Markov chain with transition probability matrix $P = (p_{ss'})$, where

$$p_{ss'} = \Pr(s_{t+1} = s' | s_t = s) > 0.$$

Suppose that the gross return on wealth is $R_{ss'} > 0$ between states s and s'. The agent is endowed with wealth $w_0 > 0$ at $t = 0$ and nothing thereafter.

(a) (4 points) Derive the budget constraint.

(b) (4 points) Let $V_s(w)$ be the value function given state s and wealth w. Derive the Bellman equation. For the expectation conditional on s, use the symbol $\mathrm{E}[\cdot \mid s]$.

(c) (4 points) Guess that the value function takes the form $V_s(w) = a_s + b_s \log w$ for some $a_s \in \mathbb{R}$ and $b_s > 0$. Assuming that the guess is correct, compute the optimal consumption rule.

(d) (4 points) Derive an equation for b_s and $\{b_{s'}\}_{s'=1}^{S}$. Letting $b = (b_1, \dots, b_S)'$ and $e = (1, \dots, 1)'$, express this equation in matrix form.

(e) (4 points) Prove that the above equation has a unique solution, and that the solution indeed satisfies $b_s > 0$ for all s.

6. This problem asks you to compute $\sqrt{3}$. Let $g(x) = x^2 - 3$. Then $x^* = \sqrt{3}$ is a solution to $g(x) = 0$.

   (a) (3 points) Suppose you use the Newton algorithm for computing x^*. Letting x_n be the approximate solution at the n-th iteration, express x_{n+1} using x_n.

   (b) (3 points) Let $x_0 = 2$. Compute x_1, x_2 (express them as **fractions**, not **decimal expansions**).

(c) (3 points) Show that $\frac{27}{16} < \sqrt{3} < \frac{7}{4}$ and $|x_1 - x^*| < 2^{-4}$.

(d) (4 points) Show that $x_{n+1} - x^* = \frac{1}{2x_n}(x_n - x^*)^2$ for all n, and also $x_n \geq x^*$.

(e) (4 points) Show that $|x_n - x^*| \leq 2^{1-5\cdot 2^{n-1}}$ for all $n \geq 1$.

(f) (3 points) Show that $|x_5 - x^*| \leq 2 \times 10^{-24}$. (Hint: $2^{10} = 1024 > 1000 = 10^3$.)

You can detach this sheet and use it as scratch paper.