Economics 205 Final Exam

Alexis Akira Toda

Fall 2018

Name: ____

Instructions:

- You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets. Calculators are not allowed.
- All logarithms are base $\mathrm{e} = 2.718281828\dots$, so $\ln x$ and $\log x$ are the same.
- Questions are not necessarily ordered in the order of difficulty, and some parts are (far) easier than others. Make sure to look at all questions and parts.

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	120
Score:							

1. Let A be an $N \times N$ positive matrix, so $A = (a_{mn})$ with $a_{mn} > 0$ for all $1 \leq m, n \leq N$. Let $\rho(A)$ be the spectral radius (largest absolute value of all eigenvalues) of A. We know from the Perron-Frobenius theorem that $\alpha = \rho(A) > 0$ is an eigenvalue of A and there exist (unique up to normalization) positive right and left eigenvectors corresponding to α, so $Ax = \alpha x$ and $y'A = \alpha y'$. Normalize the eigenvectors such that they have (Euclidean) norm 1, so

$$\|x\| = \sqrt{\sum_{n=1}^{N} x_n^2} = 1.$$

This question asks you to prove that the Perron root α and Perron vector x are smooth functions of the elements of A. Define $F : \mathbb{R}_{++}^{N^2} \times \mathbb{R}_{++}^{N} \times \mathbb{R}_{++} \to \mathbb{R}^N \times \mathbb{R}$ by

$$F(A, x, \alpha) = \begin{bmatrix} (A - \alpha I)x \\ \|x\|^2 - 1 \end{bmatrix}.$$

(a) (5 points) Compute the Jacobian $J = D_{(x,\alpha)}F$ of F with respect to (x, α).

Solution: By simple calculation, we obtain

$$J = \begin{bmatrix} A - \alpha I & -x \\ 2x' & 0 \end{bmatrix}.$$

(b) (10 points) Prove that J is regular at (A, x, α). (Hint: Assume $J \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for some $u \in \mathbb{R}^N$, $v \in \mathbb{R}$, and show that $u = 0$ and $v = 0$.)

Solution: To show that J is regular, suppose that there exist $u \in \mathbb{R}^N$ and $v \in \mathbb{R}$ such that

$$J \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A - \alpha I & -x \\ 2x' & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Comparing each block, we obtain

$$\begin{aligned} 0 &= (A - \alpha I)u - vx, \\ 0 &= x'u. \end{aligned}$$

Let $y \gg 0$ be the left Perron vector of A. Multiplying y' from left to the first equation, we obtain

$$0 = y'0 = y'(A - \alpha I)x - vy'x = -vy'x$$

because $y'A = \alpha y'$. Since $x, y \gg 0$, we have $y'x > 0$, so it must be $v = 0$. Then by the first equation we obtain $(A - \alpha I)u = 0 \iff Au = \alpha u$. Since the Perron vector is unique up to normalization, it must be $u = \lambda x$ for some $\lambda \in \mathbb{R}$. Then from the second equation we get

$$0 = x'u = \lambda x'x = \lambda \|x\|^2 = \lambda,$$

so $u = \lambda x = 0$. Therefore J is regular.

(c) (5 points) Prove that the Perron root α and Perron vector x are continuously differentiable in the elements of A.

> **Solution:** Immediate from the implicit function theorem.

2. Let
$$A = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix},$$
where $0 < p, q < 1$.

(a) (10 points) Compute all eigenvalues of A.

> **Solution:** The characteristic polynomial of A is
> $$\begin{aligned}\Phi_A(t) &= \begin{vmatrix} t-(1-p) & -p \\ -q & t-(1-q) \end{vmatrix} \\ &= t^2 - (2-p-q)t + 1 - p - q = (t-1)(t-1+p+q).\end{aligned}$$
> Therefore the eigenvalues are $t = 1, 1-p-q$.

(b) (10 points) Compute the right and left eigenvectors of A corresponding to the eigenvalue with largest absolute value. Normalize the eigenvalues such that the sum of absolute values of elements (L^1 norm) is 1.

> **Solution:** Since $0 < p, q < 1$, we have $|1-p-q| < 1$. Therefore the eigenvalue with largest absolute value is 1. The right eigenvector $x = (x_1, x_2)'$ satisfies
> $$Ax = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
> Solving the equations, we get $x_1 = x_2$, so the normalized eigenvector is $x = (1/2, 1/2)'$. Similarly, the left eigenvector $y = (y_1, y_2)'$ satisfies
> $$y'A = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}.$$
> Solving equations, we get $py_1 = qy_2$, so the normalized eigenvector is $y = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)'$.

3. Consider the optimization problem

minimize	$f(x)$
subject to	$g_i(x) \leq 0, \quad (i = 1, \dots, I)$

where $f, g_i : \mathbb{R}^N \to \mathbb{R}$ are differentiable.

(a) (4 points) What is the definition of a convex function?

Solution: f is convex if for all x_1, x_2 and $\alpha \in [0,1]$, we have

$$f((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)f(x_1) + \alpha f(x_2).$$

(b) (4 points) What is the definition of a quasi-convex function?

Solution: f is quasi-convex if for all x_1, x_2 and $\alpha \in [0,1]$, we have

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}.$$

(c) (4 points) Suppose that g_i's are convex. What is the Slater condition (constraint qualification)?

Solution: We say that the Slater condition holds if there exists some x_0 such that $g_i(x_0) < 0$ for all i.

(d) (8 points) The Karush-Kuhn-Tucker theorem (for convex functions) says that if g_i's are convex and satisfy the Slater condition, then a solution to the optimization problem satisfies the first-order and complementary slackness conditions. Is this also true if g_i's are only quasi-convex? If so, prove it. If not, provide a counterexample.

Solution: The statement is false. A counterexample is $f(x) = x$, $I = 1$, and $g(x) = -x^3$. Since g is monotone, it is quasi-convex. Furthermore, the Slater condition holds because $g(1) = -1 < 0$. Since $-x^3 \leq 0 \iff x \geq 0$, the solution is clearly $x = 0$. However, the conclusion of the KKT theorem does not hold because the first-order condition

$$0 = f'(x) + \lambda g'(x) = 1 - 3\lambda x^2$$

does not hold at the solution $x = 0$ for any $\lambda \geq 0$.

4. Consider the utility maximization problem

$$\begin{array}{ll} \text{maximize} & u(x_1, x_2) = \alpha \log x_1 + (1-\alpha)\log x_2 \\ \text{subject to} & p_1 x_1 + p_2 x_2 \leq w, \end{array}$$

where $\alpha \in (0,1)$ is a preference parameter, $x_1, x_2 > 0$ are consumption of goods 1, 2, p_1, p_2 are prices, and $w > 0$ is wealth.

(a) (10 points) Solve the utility maximization problem. Make sure that all of your arguments are rigorous.

Solution: The solution is

$$(x_1, x_2) = \left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right).$$

You need to

- show that the objective function is concave (by taking the Hessian),
- write down the Lagrangian, derive KKT conditions, and solve for the candidate, and
- mention that for concave maximization problems, KKT conditions are sufficient for optimality.

(b) (10 points) Let $V(p_1, p_2, w, \alpha)$ be the maximized utility as a function of all parameters. Derive a condition on these parameters such that liking good 1 more makes you happier, i.e., $\frac{\partial}{\partial \alpha} V > 0$.

Solution: Let

$$L(x, \lambda, p_1, p_2, w, \alpha) = \alpha \log x_1 + (1 - \alpha) \log x_2 + \lambda(w - p_1 x_1 - p_2 x_2)$$

be the Lagrangian. By the envelope theorem, we obtain

$$\frac{\partial}{\partial \alpha} V = \frac{\partial}{\partial \alpha} L = \log x_1 - \log x_2 = \log \frac{x_1}{x_2}.$$

Therefore the condition is

$$1 < \frac{x_1}{x_2} = \frac{\alpha p_2}{(1 - \alpha) p_1} \iff \alpha > \frac{p_1}{p_1 + p_2}.$$

5. Consider an infinite-horizon optimal consumption-saving problem with stochastic returns. Suppose the agent has utility function

$$\mathrm{E} \sum_{t=0}^{\infty} \beta^t \log c_t,$$

where E denotes the expectation, $\beta \in (0, 1)$ is the discount factor, and $c_t > 0$ is consumption at time t. Suppose that there are S states indexed by $s = 1, \dots, S$. The state at time t, denoted by s_t, evolves according to a Markov chain with transition probability matrix $P = (p_{ss'})$, where

$$p_{ss'} = \Pr(s_{t+1} = s' | s_t = s) > 0.$$

Suppose that the gross return on wealth is $R_{ss'} > 0$ between states s and s'. The agent is endowed with wealth $w_0 > 0$ at $t = 0$ and nothing thereafter.

(a) (4 points) Derive the budget constraint.

Solution:

$$w' = R_{ss'}(w - c).$$

(b) (4 points) Let $V_s(w)$ be the value function given state s and wealth w. Derive the Bellman equation. For the expectation conditional on s, use the symbol $\mathrm{E}[\cdot \mid s]$.

> **Solution:**
>
> $$V_s(w) = \max_c \left\{\log c + \beta \,\mathrm{E}\left[V_{s'}(R_{ss'}(w-c)) \mid s\right]\right\}.$$

(c) (4 points) Guess that the value function takes the form $V_s(w) = a_s + b_s \log w$ for some $a_s \in \mathbb{R}$ and $b_s > 0$. Assuming that the guess is correct, compute the optimal consumption rule.

> **Solution:** Substituting $V_s(w) = a_s + b_s \log w$ into the Bellman equation, we obtain
>
> $$a_s + b_s \log w = \max_c \left\{\log c + \beta \,\mathrm{E}\left[a_{s'} + b_{s'} \log(R_{ss'}(w-c)) \mid s\right]\right\}.$$
>
> The objective function is clearly concave. The first-order condition is
>
> $$\frac{1}{c} - \beta \,\mathrm{E}\left[b_{s'} \mid s\right] \frac{1}{w-c} = 0 \iff c = \frac{w}{1 + \beta \,\mathrm{E}\left[b_{s'} \mid s\right]}.$$

(d) (4 points) Derive an equation for b_s and $\{b_{s'}\}_{s'=1}^S$. Letting $b = (b_1, \dots, b_S)'$ and $e = (1, \dots, 1)'$, express this equation in matrix form.

> **Solution:** Substituting the optimal consumption rule into the Bellman equation, the coefficients of $\log w$ becomes
>
> $$b_s = 1 + \beta \,\mathrm{E}\left[b_{s'} \mid s\right] = 1 + \beta \sum_{s'=1}^{S} p_{ss'} b_{s'}.$$
>
> Putting this into a matrix, we obtain
>
> $$b = e + \beta P b.$$

(e) (4 points) Prove that the above equation has a unique solution, and that the solution indeed satisfies $b_s > 0$ for all s.

> **Solution:** One way is to note that $(I - \beta P)b = e$ implies
>
> $$b = (I - \beta P)^{-1} e = \left(\sum_{n=0}^{\infty} \beta^n P^n\right) e \gg 0.$$
>
> (Since $\rho(P) = 1$ and $\beta < 1$, we have $\rho(A) = \beta < 1$ for $A = \beta P$, so $A^n \to O$ as $n \to \infty$. Use the fact that
>
> $$I - A^{n+1} = (I - A)(I + A + \cdots + A^n)$$

to show the above result.)
Another way is to define $X = \mathbb{R}_+^S$ and $T : X \to X$ by $Tx = e + \beta P x$ and show that T is a contraction mapping. You can check this using Blackwell's sufficient condition.

6. This problem asks you to compute $\sqrt{3}$. Let $g(x) = x^2 - 3$. Then $x^* = \sqrt{3}$ is a solution to $g(x) = 0$.

   (a) (3 points) Suppose you use the Newton algorithm for computing x^*. Letting x_n be the approximate solution at the n-th iteration, express x_{n+1} using x_n.

   Solution: By the definition of the Newton algorithm, we have

   $$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{x_n^2 - 3}{2x_n} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right).$$

   (b) (3 points) Let $x_0 = 2$. Compute x_1, x_2 (express them as **fractions**, not **decimal expansions**).

   Solution: Using the above formula, we get

   $$\begin{aligned} x_1 &= \frac{1}{2}\left(2 + \frac{3}{2}\right) = \frac{7}{4}, \\ x_2 &= \frac{1}{2}\left(\frac{7}{4} + \frac{12}{7}\right) = \frac{97}{56}. \end{aligned}$$

   (c) (3 points) Show that $\frac{27}{16} < \sqrt{3} < \frac{7}{4}$ and $|x_1 - x^*| < 2^{-4}$.

   Solution: Since $(27/16)^2 = 729/256 < 3$ and $(7/4)^2 = 49/16 > 3$, we obtain $\frac{27}{16} < \sqrt{3} < \frac{7}{4}$. Since $x_1 = 7/4$ by the above calculation, it follows that

   $$|x_1 - x^*| = \left|\frac{7}{4} - \sqrt{3}\right| < \left|\frac{7}{4} - \frac{27}{16}\right| = \frac{1}{16} = 2^{-4}.$$

   (d) (4 points) Show that $x_{n+1} - x^* = \frac{1}{2x_n}(x_n - x^*)^2$ for all n, and also $x_n \geq x^*$.

   Solution: By the definition of the Newton algorithm,

   $$x_{n+1} - x^* = \frac{1}{2x_n}(x_n^2 + 3 - 2\sqrt{3}x_n) = \frac{1}{2x_n}(x_n - x^*)^2.$$

   Clearly $x_0 = 2 > \sqrt{3} = x^*$. For $n \geq 1$ we have $x_n - x^* = \frac{1}{2x_{n-1}}(x_{n-1} - x^*)^2 \geq 0$, so $x_n \geq x^*$.

   (e) (4 points) Show that $|x_n - x^*| \leq 2^{1 - 5 \cdot 2^{n-1}}$ for all $n \geq 1$.

Solution: Since

$$|x_1 - x^*| \le 2^{-4} = 2^{1-5\cdot 2^{1-1}},$$

the claim is true for $n = 1$. Assume that the claim is true for some n. Since $x_n \ge x^* = \sqrt{3} > 1$, we obtain

$$|x_{n+1} - x^*| = \frac{1}{2x_n}\,|x_n - x^*|^2 \le \frac{1}{2}(2^{1-5\cdot 2^{n-1}})^2 = \frac{1}{2}2^{2-5\cdot 2^n} = 2^{1-5\cdot 2^n},$$

so the claim holds for $n + 1$ as well. Therefore by mathematical induction, the claim holds for all $n \ge 1$.

(f) (3 points) Show that $|x_5 - x^*| \le 2 \times 10^{-24}$. (Hint: $2^{10} = 1024 > 1000 = 10^3$.)

Solution: By the above question,

$$|x_5 - x^*| \le 2^{1-5\cdot 2^4} = 2 \times 2^{-80} < 2 \times (10^{-3})^8 = 2 \times 10^{-24}.$$

You can detach this sheet and use it as scratch paper.