Economics 205 Final Exam

Alexis Akira Toda

Fall 2019

Name: ____

Instructions:

- You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets. Calculators are not allowed.
- All logarithms are base $\mathrm{e} = 2.718281828\dots$, so $\ln x$ and $\log x$ are the same.
- Questions are not necessarily ordered according to difficulty, and some parts are (far) easier than others. Make sure to look at all questions and parts before deciding the order to solve.

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	120
Score:							

1. (a) (5 points) Let A be a square matrix. Show that

$$(I - A)(I + A + \cdots + A^{k-1}) = I - A^k.$$

(b) (15 points) Let A be a nonnegative square matrix. Show that if $z > \rho(A)$, then the matrix $zI - A$ is regular and $(zI - A)^{-1}$ is nonnegative. (Here $\rho(A)$ is the spectral radius of A, which is the largest absolute value of all eigenvalues.)

2. Let $C \subset \mathbb{R}^N$ be a convex set. A nonnegative function $f : C \to \mathbb{R}$ is called *log-convex* if either (i) $f(x) = 0$ for all $x \in C$ or (ii) $f(x) > 0$ for all $x \in C$ and $\log f$ is convex. Prove the followings.

(a) (5 points) If f is log-convex, then f^λ is log-convex for all $\lambda \geq 0$.

(b) (5 points) If f, g are log-convex, then fg is log-convex.

(c) (10 points) If f, g are log-convex, then $f + g$ is log-convex. (Hint: you may use Hölder's inequality

$$\sum_{n=1}^{N} u_n v_n \leq \left(\sum_{n=1}^{N} u_n^p \right)^{1/p} \left(\sum_{n=1}^{N} v_n^q \right)^{1/q},$$

where $u_n \geq 0$, $v_n \geq 0$, and $p, q > 1$ are numbers such that $1/p + 1/q = 1$.)

3. (a) (5 points) Let $M_N(\mathbb{R})$ be the set of real $N \times N$ matrices. What is the definition of a matrix norm on $M_N(\mathbb{R})$?

(b) (5 points) For $A = (a_{mn}) \in M_N(\mathbb{R})$, define $\|A\| = \sqrt{\sum_{m=1}^{N} \sum_{n=1}^{N} a_{mn}^2}$. Prove that $\|\cdot\|$ is a matrix norm.

(c) (10 points) Let C be a convex set. Let $A(x) = (a_{mn}(x))$ be a nonnegative matrix, where each $a_{mn} : C \to \mathbb{R}$ is log-convex. (See the previous problem for the definition of a log-convex function.) Let $f(x) = \rho(A(x))$, where ρ is the spectral radius. Show that f is log-convex. (Hint: you may use the property $\rho(A) = \lim_{k\to\infty} \left\|A^k\right\|^{1/k}$.)

4. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0 \quad (i = 1, \ldots, I), \end{array}$$

where $f, g_i : \mathbb{R}^N \to \mathbb{R}$ are differentiable. Let $C = \left\{ x \in \mathbb{R}^N \,\middle|\, (\forall i) g_i(x) \leq 0 \right\}$ be the constraint set and $\bar{x} \in C$.

(a) (5 points) Define the tangent cone of C at $\bar{x}$, denoted by $T_C(\bar{x})$.

(b) (5 points) Define the linearizing cone of C at $\bar{x}$, denoted by $L_C(\bar{x})$.

(c) (10 points) Define the Guignard constraint qualification. Prove that when all constraints are linear, so $g_i(x) = \langle a_i, x\rangle - c_i$ for some $0 \neq a_i \in \mathbb{R}^N$ and $c_i \in \mathbb{R}$, then the Guignard constraint qualification holds.

5. Consider the problem

$$\begin{aligned} &\text{maximize} && \frac{4}{3}x_1^3 + \frac{1}{3}x_2^3 \\ &\text{subject to} && x_1 + x_2 \leq 1, \\ & && x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(a) (5 points) Are the Karush-Kuhn-Tucker conditions necessary for a solution? Answer yes or no, then explain why.

(b) (5 points) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? Answer yes or no, then explain why.

(c) (5 points) Write down the Lagrangian.

(d) (5 points) Compute the solution.

6. (20 points) Choose one of the following theorems and state it as precisely as possible:

- separating hyperplane theorem
- contraction mapping theorem
- implicit function theorem

You can detach this sheet and use it as scratch paper.