Robust Asset-Liability Management[1]

Tjeerd de Vries Alexis Akira Toda

Department of Economics
University of California San Diego

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[1]Link to paper: https://arxiv.org/abs/2310.00553

Hedging interest rate risk

- Many financial institutions have long-term commitments
  - Insurance companies: promise insurance payments
  - Pension funds: promise (defined-benefit) pensions
  - Banks: fund long-term projects with deposits
- Asset-liability management: cover future liabilities by holding sufficient assets
  - Old problem, but still relevant (e.g., collapse of Silicon Valley Bank)
- If market complete, problem trivial by replicating liabilities with zero-coupon bonds (dedication)
- In practice, maturity of liabilities could exceed longest maturity of government bonds, so market incomplete and can only approximate (immunization)

Bond price and duration

- Consider cash flows $f_1, \dots, f_N$ paid out at time $t_1, \dots, t_N$
- Assuming constant interest rate r, present value is

$$P = \sum_{n=1}^{N} \mathrm{e}^{-rt_n} f_n$$

- Interest rate sensitivity of bond price is

$$D := -\frac{\partial \log P}{\partial r} = -\frac{1}{P}\frac{\partial P}{\partial r} = \frac{1}{P}\sum_{n=1}^{N} t_n \mathrm{e}^{-rt_n} f_n$$

- D is called duration because it is weighted average of time to payment: $D = \sum_{n=1}^{N} w_n t_n$ for $w_n := \mathrm{e}^{-rt_n} f_n / P$, with $\sum w_n = 1$

Classical immunization

- Interest rate need not be constant; if $y(t)$ denote (pure) yield at term t, bond price and duration are

$$P = \sum_{n=1}^{N} \mathrm{e}^{-y(t_n)t_n} f_n,$$

$$D = \frac{1}{P} \sum_{n=1}^{N} t_n \mathrm{e}^{-y(t_n)t_n} f_n$$

- Here duration is sensitivity of bond price with respect to parallel shift in yield curve Example
- Classical immunization prescribes to match duration of asset and liability so that equity (asset minus liability) is insensitive to yield curve shifts (Macaulay, 1938; Samuelson, 1945; Redington, 1952)

Limitations of classical immunization

- By construction, only allows for parallel shifts to yield curve, but in practice yield curve can change in many ways
- Because duration is only a scalar, when there are many bonds, it is not obvious how to choose portfolio (indeterminacy)
- Generalizations have been proposed, for instance matching convexity (high-order duration)

$$\frac{1}{P}\sum_{n=1}^{N} t_n^2 \mathrm{e}^{-y(t_n)t_n} f_n,$$

  but it has been found to lead to portfolio instability and poor performance
- Many other ad hoc methods but lack of principle

This paper

- Propose new robust immunization method that maximizes equity against arbitrary interest rate shock
- Idea: span perturbations to yield curve by basis functions, and optimize against worst case perturbation
- Tools:
  - functional analysis: Gateaux derivative
  - numerical analysis: Chebyshev polynomials
- Excellent performance in static and dynamic hedging experiments

Model

- Continuous time, $t \in [0, T]$
- J available bonds for trade; bond j's cumulative payout denoted by weakly increasing $F_j : [0, T] \to \mathbb{R}_+$
  - If zero-coupon bond with face value 1 and maturity t_j, then

$$F_j(t) = \begin{cases} 0 & \text{if } 0 \le t < t_j, \\ 1 & \text{if } t_j \le t \le T \end{cases}$$

  - If continuously pay out coupon c_j, then $F_j(t) = c_j t$
- $F : [0, T] \to \mathbb{R}_+$: cumulative cash flow to be immunized
- $y : [0, T] \to \mathbb{R}$: yield curve
- Present value of liability is Riemann-Stieltjes integral

$$\int_0^T \mathrm{e}^{-ty(t)} \, \mathrm{d}F(t)$$

Cumulative discount rate

- Convenient to define "cumulative discount rate" $x(t) := ty(t)$
- By definition of forward rate, we have

$$x(t) = \int_0^t f(u) \, \mathrm{d}u,$$

  where $f(u)$ is instantaneous forward rate at term u
- Present value of asset/liability becomes functional

$$P(x) := \int_0^T \mathrm{e}^{-x(t)} \, \mathrm{d}F(t),$$

- Fund manager's problem is to choose bond portfolio $z = (z_j) \in \mathbb{R}^J$ to approximate $P(x)$ by $\sum_{j=1}^J z_j P_j(x)$ in some optimal way

Robust immunization problem

- $\mathcal{Z} \subset \mathbb{R}^J$: set of admissible portfolios
- $\mathcal{H}$: set of admissible perturbations to cumulative discount rate
- After perturbation $h \in \mathcal{H}$, portfolio value ("asset") is

$$V(z, x+h) := \sum_{j=1}^{J} z_j P_j(x+h)$$

- Hence asset minus liability ("equity") is

$$E(z, x+h) := V(z, x+h) - P(x+h)$$

- Fund manager seeks to maximize worst case equity, so solve

$$\sup_{z \in \mathcal{Z}} \inf_{h \in \mathcal{H}} E(z, x+h)$$

Assumptions

Assumption (Discrete payouts)

The bonds and liability pay out on finitely many dates, whose union is denoted by $\{t_n\}_{n=1}^N \subset (0, T]$.

Assumption (Portfolio constraint)

The set of admissible portfolios $\mathcal{Z} \subset \mathbb{R}^J$ *is nonempty and closed. Furthermore, all* $z \in \mathcal{Z}$ *satisfy value matching:*

$$P(x) = \sum_{j=1}^{J} z_j P_j(x).$$

- Merely a normalization (objective function $= 0$ at $h = 0$)

Space of cumulative discount rates

- Let $C^r[0, T]$ be vector space of r-times continuously differentiable functions on $[0, T]$
- Space of forward rates is $C[0, T]$ endowed with supremum norm $\|f\|_\infty = \sup_{t\in[0,T]} |f(t)|$
- Since cumulative discount rate is integral of forward rate, $x(t) = \int_0^t f(u)\,\mathrm{d}u$, let

$$\mathcal{X} = \left\{x \in C^1[0, T] : x(0) = 0\right\}$$

- $\mathcal{X}$ is Banach space with norm $\|x\|_\mathcal{X} = \sup_{t\in[0,T]} |x'(t)|$

Assumption

Assumption (Basis)

There exists a countable basis $\{h_i\}_{i=1}^{\infty}$ of $\mathcal{X}$ such that for each $I \in \{1, \ldots, N\}$, the $I \times N$ matrices $H = (h_i(t_n))$ and $G = (h_i'(t_n))$ have full row rank.

- This assumption allows us to
  - approximate any $x \in \mathcal{X}$ by finitely many basis functions, and
  - avoid indeterminacy
- Example: if h_i polynomial of degree i with $h_i(0) = 0$, then OK by Stone-Weierstraß theorem
- Intuitively, H (G) is matrix of perturbations to cumulative discount rate (forward rate) evaluated at payout dates

Admissible perturbations

- To operationalize, define set of admissible perturbations by

$$\mathcal{H}_I(\Delta) := \left\{ h \in \operatorname{span}\{h_i\}_{i=1}^I : (\forall n) \left| h'(t_n) \right| \leq \Delta \right\}$$

- Intuition: perturb forward rate by at most $\pm\Delta$ within span of first I basis functions
- Thus final maxmin problem is

$$\sup_{z \in \mathcal{Z}} \inf_{h \in \mathcal{H}_I(\Delta)} E(z, x + h)$$

- Note: setting in classical immunization corresponds to $I = 1$ and $h_1(t) = t$ (hence $h_1'(t) = 1$)

Gateaux and Fréchet derivatives

- For perturbation $h \in \mathcal{X}$, rate of change in liability value is Gateaux derivative

$$\delta P(x;h) \coloneqq \lim_{\alpha \to 0} \frac{1}{\alpha}(P(x+\alpha h) - P(x))$$
$$= -\int_0^T \mathrm{e}^{-x(t)} h(t)\,\mathrm{d}F(t)$$

- Can define bounded linear operator $P'(x)$ by

$$P'(x)h = -\int_0^T \mathrm{e}^{-x(t)} h(t)\,\mathrm{d}F(t),$$

called Fréchet derivative

Sensitivity matrix/vector

- Define sensitivity matrix $A = (a_{ij}) \in \mathbb{R}^{I \times J}$ and vector $b = (b_i) \in \mathbb{R}^I$ with respect to basis functions by

$$a_{ij} := -\frac{\delta P_j(x; h_i)}{P(x)} = \frac{1}{P(x)} \int_0^T \mathrm{e}^{-x(t)} h_i(t) \, \mathrm{d}F_j(t),$$

$$b_i := -\frac{\delta P(x; h_i)}{P(x)} = \frac{1}{P(x)} \int_0^T \mathrm{e}^{-x(t)} h_i(t) \, \mathrm{d}F(t)$$

- Note: b_i is duration if $h_i(t) = t$
- Convenient to define $A_+ \in \mathbb{R}^{(I+1) \times J}$ and $b_+ \in \mathbb{R}^{I+1}$ by

$$A_+ := \begin{bmatrix} a_0 \\ A \end{bmatrix} \quad \text{and} \quad b_+ := \begin{bmatrix} b_0 \\ b \end{bmatrix},$$

where a_{0j}, $b_0 = 1$ defined analogously using $h_0(t) \equiv 1$

Sensitivity of equity

- Recall definition of equity

$$E(z,x) := \sum_{j=1}^{J} z_j P_j(x) - P(x)$$

- If perturbation is $h = \Delta \sum_{i=1}^{I} w_i h_i \in \mathcal{H}_I(\Delta)$, sensitivity of equity becomes

$$\lim_{\Delta \to 0} \frac{1}{\Delta P(x)} E(z, x+h) = -\langle w, Az - b \rangle$$

- For $h \in \mathcal{H}_I(\Delta)$, coefficients $w = (w_i)$ need to satisfy certain restrictions

Auxiliary problem

- Straightforward to show $h = \Delta \sum_{i=1}^{I} w_i h_i \in \mathcal{H}_I(\Delta)$ if and only if

$$\mathcal{W} := \left\{ w \in \mathbb{R}^I : G' w \in [-1, 1]^N \right\},$$

where $G = (h_i'(t_n)) \in \mathbb{R}^{I \times N}$

- This motivates solving auxiliary problem

$$\sup_{z \in \mathcal{Z}} \inf_{w \in \mathcal{W}} -\langle w, Az - b \rangle \iff \inf_{z \in \mathcal{Z}} \sup_{w \in \mathcal{W}} \langle w, Az - b \rangle$$

to solve maxmin problem

$$\sup_{z \in \mathcal{Z}} \inf_{h \in \mathcal{H}_I(\Delta)} E(z, x + h)$$

Solution to auxiliary problem

Proposition (Minmax)

Suppose Assumptions hold, $I \geq J-1$, and sensitivity matrix A_+ has full column rank. Then

1. *There exists $(z^*, w^*) \in \mathcal{Z} \times \mathcal{W}$ that achieves minmax value*

$$V_I(\mathcal{Z}) := \inf_{z \in \mathcal{Z}} \sup_{w \in \mathcal{W}} \langle w, Az - b \rangle .$$

2. *$V_I(\mathcal{Z}) \geq 0$, and $z \in \mathcal{Z}$ achieves $V_I(\mathcal{Z}) = 0$ if and only if $A_+ z = b_+$.*

Robust immunization

Theorem (Robust immunization)

1. *Guaranteed equity satisfies*

$$\lim_{\Delta\downarrow 0}\frac{1}{\Delta}\sup_{z\in\mathcal{Z}}\inf_{h\in\mathcal{H}_I(\Delta)}E(z,x+h)=-P(x)V_I(\mathcal{Z}).$$

2. *Letting $z^*\in\mathcal{Z}$ be solution to auxiliary problem and $\theta_j:=z_jP_j(x)/P(x)$ be corresponding portfolio share, then*

$$\sup_{h\in\mathcal{H}_I(\Delta)}|E(z^*,x+h)|\le\Delta P(x)\left(V_I(\mathcal{Z})+\frac{1}{4}\Delta T^2\mathrm{e}^{\Delta T}(1+\|\theta\|_1)\right).$$

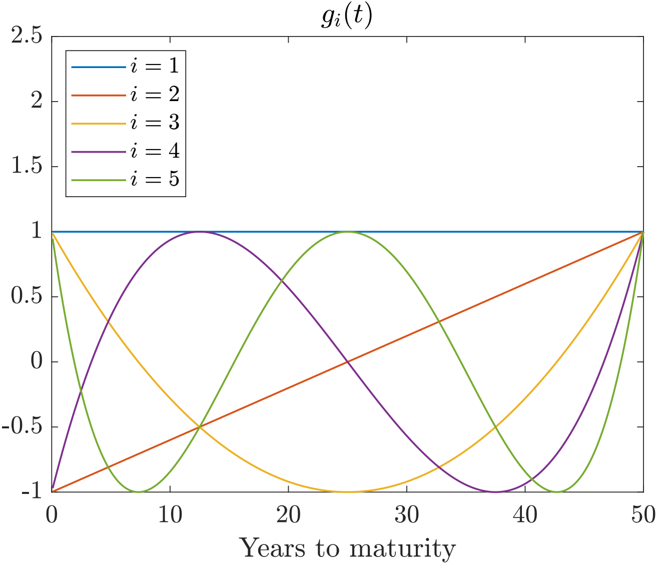
- Solution z^* to auxiliary problem achieves guaranteed equity in limit $\Delta\downarrow 0$

Implementation

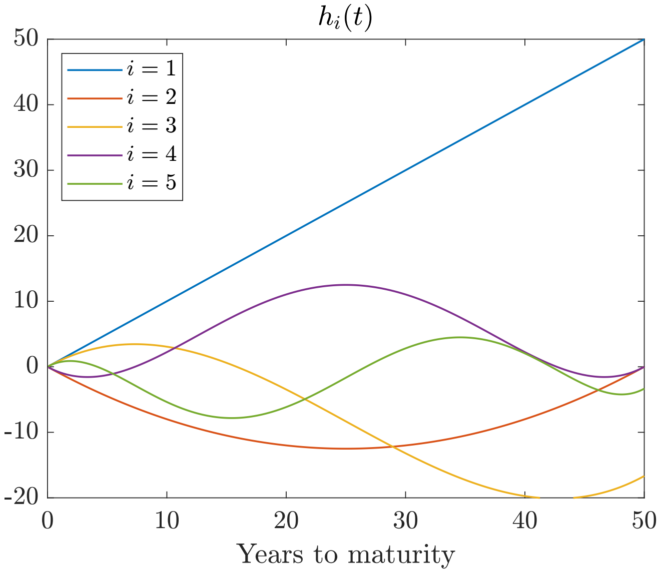
- Implementation requires choice of basis functions $\{h_i\}_{i=1}^{\infty}$
- Natural to use (Chebyshev) polynomials because Assumptions satisfied and good approximation property (Trefethen, 2019)
- Let $T_n : [-1, 1] \to \mathbb{R}$ be Chebyshev polynomial defined by $T_n(\cos\theta) = \cos n\theta$
- Map $[-1, 1]$ to $[0, T]$ by affine transformation, so let $g_i(t) = T_{i-1}(2t/T - 1)$ be basis for forward rate
- Define basis for cumulative discount rate by

$$h_i(t) = \int_0^t g_i(u)\,\mathrm{d}u$$

Basis for forward rate



Basis for cumulative discount rate



How good is forward rate approximation?

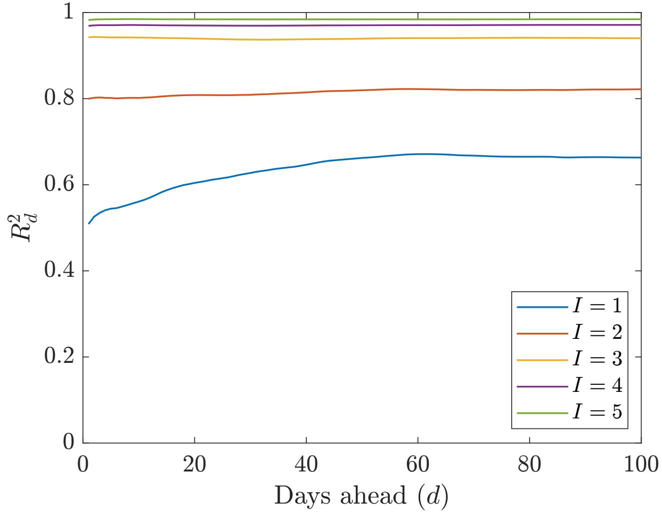
- Use 1985–2022 daily yield curve data from Gürkaynak et al. (2007), who estimate Svensson (1994) model
- For each day s and term $t_n = n/12$ ($n = 1, \dots, 360$), calculate the d-day ahead forward rate change $f_{s+d}(t_n) - f_s(t_n)$
- For each s and $d = 1, \dots, 100$, run OLS regression

$$f_{s+d}(t_n) - f_s(t_n) = \sum_{i=1}^{I} \gamma_{isd} g_i(t_n) + \epsilon_s(t_n), \quad n = 1, \dots, N$$

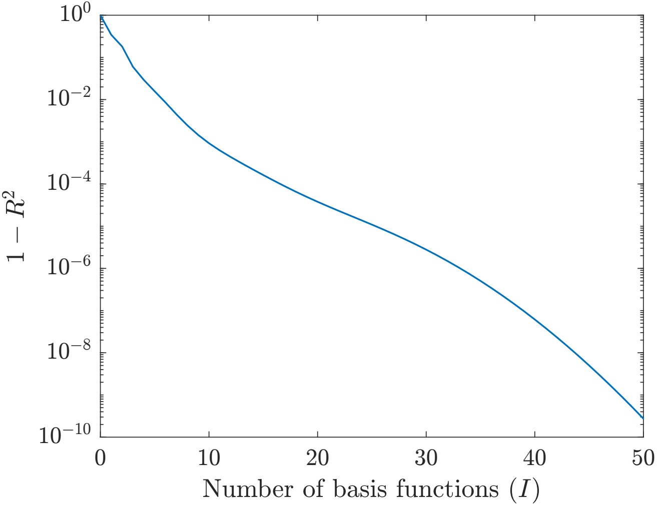
- Calculate goodness-of-fit measure

$$R_d^2 := \frac{\sum_{s=1}^{S} \sum_{n=1}^{N} \left(\sum_{i=1}^{I} \widehat{\gamma}_{isd} g_i(t_n) \right)^2}{\sum_{s=1}^{S} \sum_{n=1}^{N} (f_{s+d}(t_n) - f_s(t_n))^2}$$

Goodness-of-fit



Goodness-of-fit



Implementing robust immunization

1. Let
   - $\mathbf{t} = (t_1, \dots, t_N)$ be $1 \times N$ vector of asset/liability payout dates
   - $\mathbf{y} = (y_1, \dots, y_N)$ be $1 \times N$ vector of yields
   - $\mathbf{f} = (f_1, \dots, f_N)$ be $1 \times N$ vector of liabilities
   - $\mathbf{F} = (f_{jn})$ be $J \times N$ matrix of bond payouts
2. Let $I \geq J - 1$, define basis functions $\{h_i\}$ as above, construct matrices of
   - basis functions $\mathbf{H} = (h_i(t_n)) \in \mathbb{R}^{I \times N}$,
   - derivatives $\mathbf{G} = (h_i'(t_n)) = (g_i(t_n)) \in \mathbb{R}^{I \times N}$,
   - zero-coupon bond prices $\mathbf{p} = \exp(-\mathbf{y} \odot \mathbf{t}) \in \mathbb{R}^{1 \times N}$
3. Define sensitivity matrix/vectors by

$$A := (\mathbf{H}\,\mathrm{diag}(\mathbf{p})\mathbf{F}')/(\mathbf{p}\mathbf{f}'),$$
$$b := \mathbf{H}\,\mathrm{diag}(\mathbf{p})\mathbf{f}'/(\mathbf{p}\mathbf{f}')$$

Implementing robust immunization

4. Let

$$\mathcal{W} := \left\{ w \in \mathbb{R}^I : G'w \in [-1, 1]^N \right\}$$

and solve auxiliary problem

$$V_I(\mathcal{Z}) := \inf_{z \in \mathcal{Z}} \sup_{w \in \mathcal{W}} \langle w, Az - b \rangle$$

- Inner maximization large linear programming problem, so easy to solve
- Outer minimization small convex minimization problem, so easy to solve

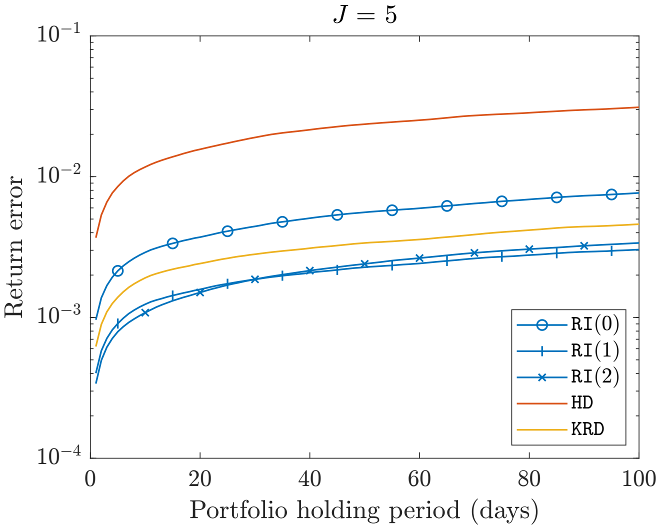
Static hedging

- Liability: constant monthly cash flow with maturity 50 years
- Asset: portfolio of zero-coupon bonds with maturity 1, 2, 5, 10, 30 years
- For static hedging, use daily yield curve data as before, and assume yield curve instantaneously changes to d-day ahead
- Evaluate performance on day s by relative return error

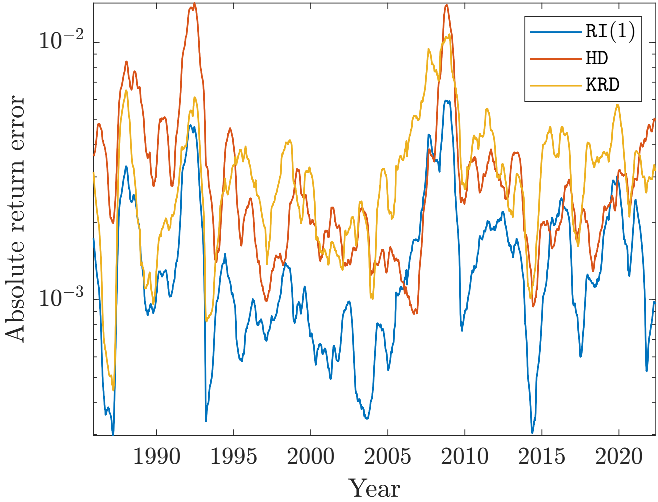
$$\frac{1}{P(x_s)}\left|\sum_{j=1}^{J} z_{sj}P_j(x_{s+d}) - P(x_{s+d})\right|$$

- Compare robust immunization (RI(0,1,2)) ▸ ? , high-order duration (HD), and key rate duration (KRD) of Ho (1992)

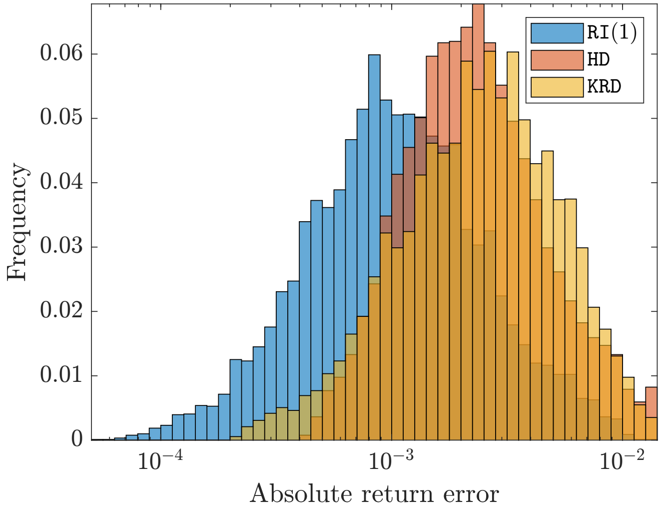
Average *d*-day ahead error



Time series of 30-day ahead error

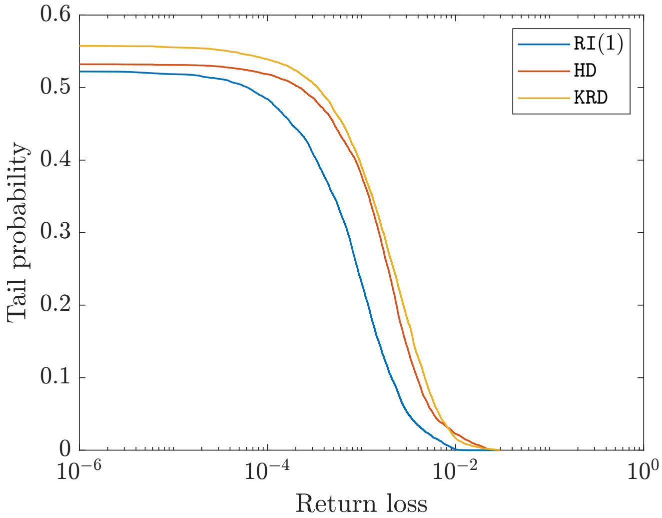


Histogram

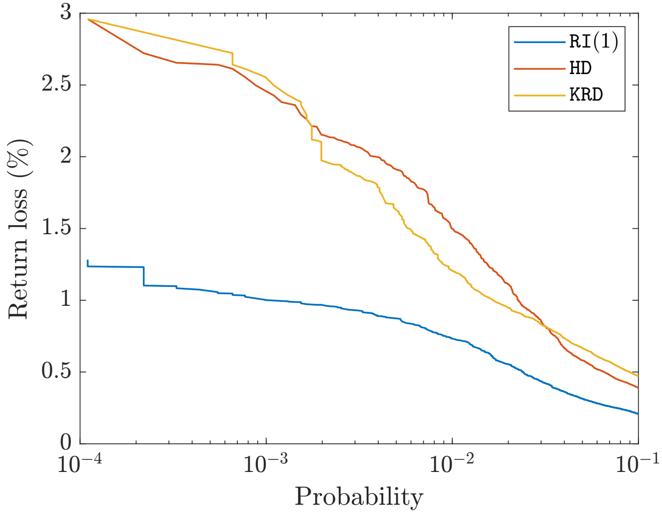


0.06
0.05
0.04
0.03
0.02
0.01
0
Frequency
10^{-4}
10^{-3}
10^{-2}
Absolute return error
RI(1)
HD
KRD

Tail probability



Value at risk



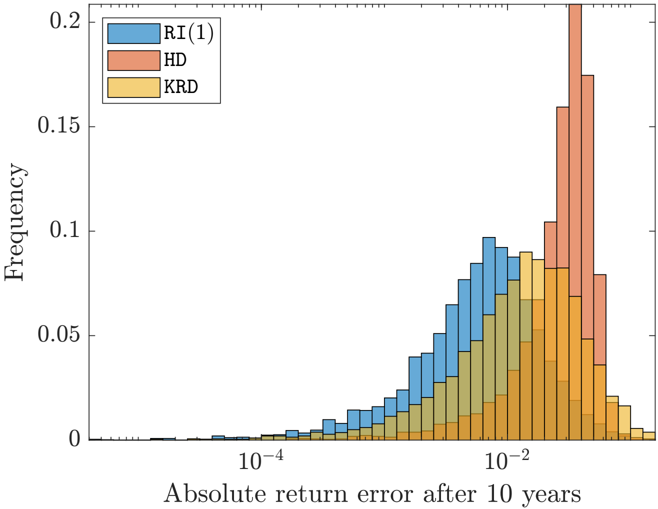
Dynamic hedging

- Estimate no-arbitrage model of Ang et al. (2008) by maximum likelihood
- Generate simulated data to evaluate dynamic hedging
- Let $s = \Delta, 2\Delta, \dots$ be rebalancing date ($\Delta =$ one quarter)
- Net asset value (NAV) of fund at date s is (with $s^{-} = s - \Delta$)

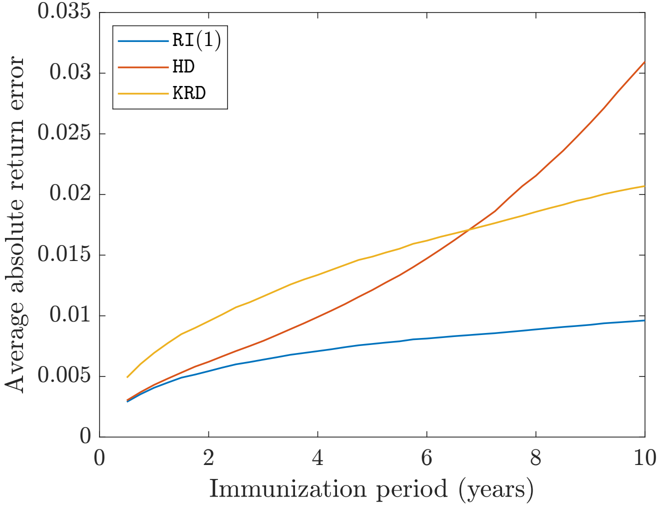
$$V_s := \underbrace{R_{s^-} C_{s^-}}_{\text{cash}} + \underbrace{\sum_{j=1}^{J} z_{s^- j} \mathrm{e}^{-x_s (t_j - \Delta)}}_{\text{bond}} - \underbrace{f_s}_{\text{liability}}$$

- Can show dynamic hedging reduces to static hedging except reducing maturities by Δ everywhere

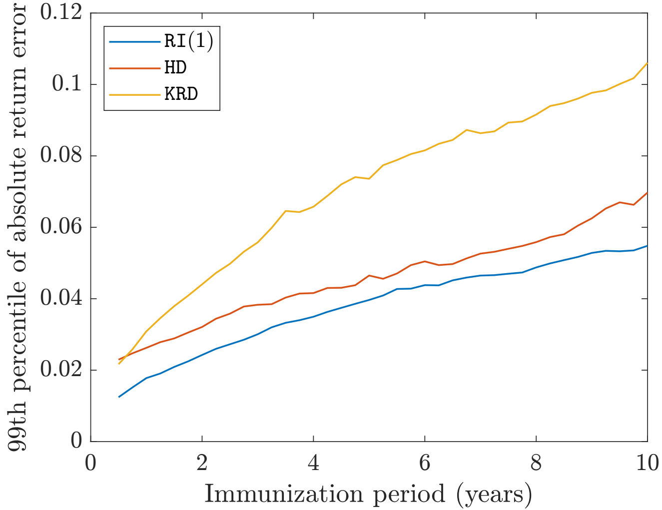
Histogram of 10-year return error



Average return error



99th percentile of return error



Conclusion

- When the world is complicated, it is natural to optimize against the worst case
- Robust immunization maximizes equity (asset minus liability) against worst interest rate shock
- Easy to implement, excellent performance

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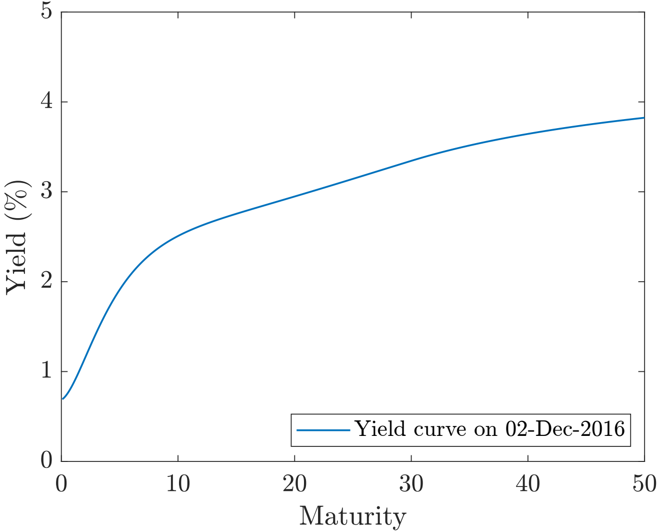
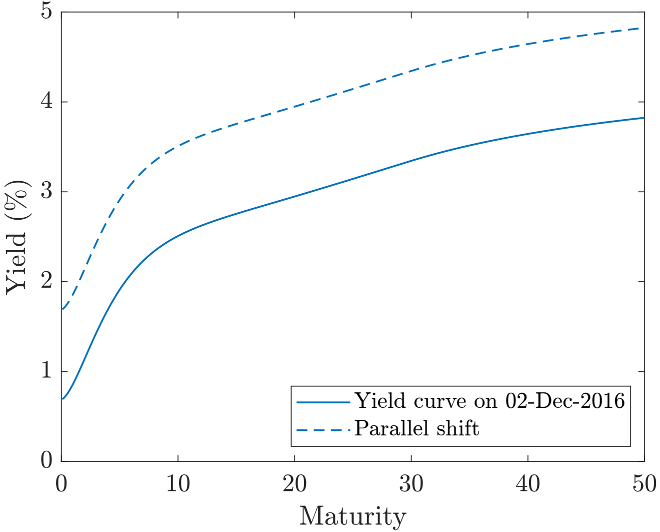
Yield curve and parallel shift



Yield curve and parallel shift

Return



Robust immunization with principal components

- Suppose perturbations to forward rates have factor structure
  - E.g., parallel shift is dominant
- For $\Delta_1 \gg \Delta_2 > 0$, consider set of admissible perturbations

$$\mathcal{H}_I(\Delta_1, \Delta_2) \\ := \left\{ h : (\exists \alpha)(\forall n) \left| \alpha h_1'(t_n) \right| \leq \Delta_1, \left| h'(t_n) - \alpha h_1'(t_n) \right| \leq \Delta_2 \right\}$$

- Idea:
  - First, perturb forward rate in direction h_1' by magnitude at most Δ_1,
  - Then perturb in arbitrary direction by magnitude at most Δ_2

Robust immunization with principal components

Theorem

Suppose the set

$$\mathcal{Z}_1 := \left\{ z \in \mathcal{Z} : \sum_{j=1}^{J} a_{1j} z_j = b_1 \right\}$$

is nonempty. Then guaranteed equity satisfies

$$\lim \frac{1}{\Delta_2} \sup_{z \in \mathcal{Z}} \inf_{h \in \mathcal{H}_I(\Delta_1, \Delta_2)} E(z, x+h) = -P(x) V_I(\mathcal{Z}_1),$$

where the limit is taken over $\Delta_1, \Delta_2 \to 0$, $\Delta_1/\Delta_2 \to \infty$, *and* $\Delta_1^2/\Delta_2 \to 0$.