A General Equilibrium Model of Satisficing Behavior

Alexis Akira Toda

Department of Economics, Yale University

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Motivation

- Walrasian equilibrium is defined by
  1. agent optimization,
  2. rational (correct) expectations,
  3. market clearing.
- However, ordinary people (including myself) do not seem to optimize but only "satisfice".
- Can we build a GE model of satisficing behavior?

This paper

- Follow-up paper of Foley "A Statistical Equilibrium Model of Markets" (JET 1994) and Toda (ET, 2010).
- Model "satisficing behavior" by prior distributions on actions; define equilibrium by posterior distributions conditional on market clearing with correct expectations.
- Prove
  1. existence of equilibrium,
  2. "informational efficiency" of equilibrium,
  3. Walrasian equilibrium is a special case of Bayesian general equilibrium.

Model

- $i \in I = \{1, 2, \dots, I\}$: agent types.
- Continuum of agents of type i with mass $n_i > 0$, where $\sum n_i = 1$.
- $e_i \in \mathbb{R}^C_+$: endowment of type i.
- $p \in \mathbb{R}^C_+$, $p_1 + p_2 + \cdots + p_C = 1$: price. (Also denoted $p \in \Delta^{C-1}$.)

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- $\mu_{i,p}$: prior probability measure on $\mathbb{R}^C_+$. Represents type i's satisficing behavior, given price.

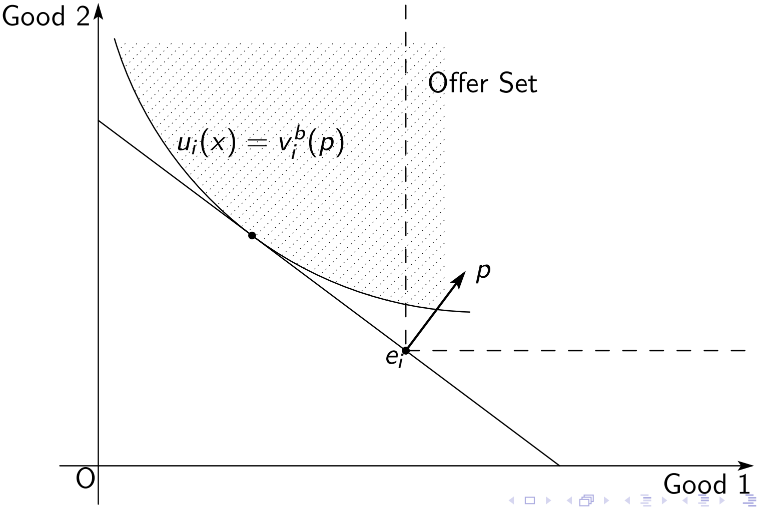
Economy is defined by

$$\mathcal{E} = \{I, \{n_i\}, \{e_i\}, \{\mu_{i,p}\}\}.$$

Offer set

- $X_{i,p} = \operatorname{supp} \mu_{i,p}$: type i's offer set at p.
- If $X_{i,p}$ consists of the Walrasian demand, then $X_{i,p}$ (as a function of p) is offer curve. Hence the name offer set.
- In this paper I do not model $\mu_{i,p}$ (or $X_{i,p}$): these are primitives. Existence of equilibrium does not depend on the specifics of $\mu_{i,p}$.

Example



Temporary (partial) equilibrium

- Since markets must clear in equilibrium, the most natural way to define temporary (partial) equilibrium is by posterior densities (f_i) conditional on market clearing.
- That is, find posterior densities with

$$\bar{x}[(f_i);(\mu_{i,p})] := \sum_{i=1}^{I} n_i \int x f_i(x)\mu_{i,p}(\mathrm{d}x) \leq \sum_{i=1}^{I} n_i e_i =: \bar{e}.$$

- But how to compute posterior densities (f_i)?
- Answer: minimize Kullback-Leibler information (relative entropy).

Entropy, Kullback-Leibler information

- Shannon entropy of multinomial distribution $\mathbf{p} = (p_1, \dots, p_K)$ is $H(\mathbf{p}) = -\sum p_k \log p_k$.
- Kullback-Leibler information (relative entropy) of posterior $\mathbf{p}$ w.r.t. prior $\mathbf{q}$ is $H(\mathbf{p}; \mathbf{q}) = \sum p_k \log(p_k/q_k)$. Hence entropy is K-L info w.r.t. uniform distribution modulo sign and additive constant.
- K-L info has natural generalization to any measure space, where as not entropy. If μ is a reference measure and $P, Q \ll \mu$ with densities p, q, then

$$H(P; Q) = \int p \log \frac{p}{q} \mathrm{d}\mu.$$

Bayes rule implies minimum K-L info

Theorem (van Campenhout & Cover, 1981)

Suppose $\{X_n\}$ *i.i.d. with prior density* g*. Then as* $N \to \infty$ *the posterior density conditional on the sample moment constraint* $\frac{1}{N}\sum_{n=1}^{N} T(X_n) = \bar{T}$ *converges to the minimum K-L information density*

$$\arg\min_f H(f;g) \text{ s.t. } \int T(x)f(x)\mathrm{d}x = \bar{T}$$

- Hence in large samples, Bayes rule implies minimum K-L info.
- Csiszár (1984) generalizes to convex inequality constraints.

Solving for temporary equilibrium

- In our case K-L info is

$$H[(f_i);(\mu_{i,p})] := \sum_{i=1}^{I} n_i \int f_i \log f_i \mathrm{d}\mu_{i,p}.$$

- Hence temporary equilibrium is the (unique) solution to

$$\min_{(f_i)} H[(f_i);(\mu_{i,p})] \text{ s.t. } \bar{x}[(f_i);(\mu_{i,p})] \leq \bar{e}.$$

- Let $\pi \in \mathbb{R}_+^C$ be Lagrange multiplier. Since π is a shadow price of commodities, it gives a natural price updating rule: set next price $\propto \pi$.

Solving for temporary equilibrium

Theorem (Existence, uniqueness, duality)

Let

$$H^*(\xi) = \sum_{i=1}^{I} n_i \left[\xi' e_i + \log \left(\int \mathrm{e}^{-\xi' x} \mu_{i,p}(\mathrm{d}x) \right) \right].$$

If

$$\left(\sum_{i=1}^{I} n_i (\operatorname{co} X_{i,p} - e_i) \right) \cap (-\mathbb{R}^C_{++}) \neq \emptyset,$$

then there exists a unique temporary equilibrium $f = (f_i)$ *and*

$$\min \{ H[f; \mu_p] \mid \bar{x}[f; \mu_p] \leq \bar{e} \} = - \min_{\xi \in \mathbb{R}^C_+} H^*(\xi).$$

Correct expectations equilibrium

- In general equilibrium, agents must correctly anticipate their trading environment (price) and behavior (posterior).
- First condition is met if $\pi \propto p$; second is met if posterior solves min K-L problem. Therefore:

Definition (Correct expectations equilibrium)

Price $p \in \Delta^{C-1}$ and densities $f = (f_i)$ constitute a correct expectations equilibrium if

1. $f = (f_i)$ is the posterior conditional on market clearing: f solves
$$\min H[f; \mu_p] \text{ subject to } \bar{x}[f; \mu_p] \leq \bar{e},$$
2. the Lagrange multiplier is proportional to p.

Equilibrium

Definition (Degenerate equilibrium)

A price $p \in \Delta^{C-1}$ and points $x_i \in \operatorname{co} X_{i,p}$ are called a degenerate equilibrium if

1. $\sum_{i=1}^{I} n_i(x_i - e_i) \leq 0$,
2. For all i, $p'(x - e_i) \geq 0$ for all $x \in X_{i,p}$.

- Need to consider the degenerate case to allow distributions to concentrate on some points (density like Dirac delta function).
- In this case all transactions must have nonnegative value, otherwise can decrease K-L info by spreading density.

Assumptions

A (degenerate or non-degenerate) correct expectations equilibrium exists under reasonable (and very weak) assumptions.

A1 (Finite measures)

For all agent types i and price p, the measure $\mu_{i,p}$ is finite.

- Note that $\mu_{i,p}$ is a prior, so $\mu_{i,p}(\mathbb{R}^C) = 1$.

Assumptions

A2 (Budget feasibility)

For all agent types i and price p, we have

$$\inf \left\{ p'(x - e_i) \,\middle|\, x \in X_{i,p} \right\} \leq 0.$$

- Offer set $X_{i,p} = \operatorname{supp} \mu_{i,p}$ are those transactions that type i agents expect to engage with positive probability.
- $p'(x - e_i) \leq 0$ for some transactions implies that agents are realistic: agents put some probability on trades within their budget.

Assumptions

A3 (Continuity of measure)

The mapping $p \mapsto \mu_{i,p}$ is weakly continuous, *i.e.*, for every sequence $\{p_n\}$ such that $p_n \to p$ and bounded measurable function f, we have

$$\lim_{n\to\infty} \int f \mathrm{d}\mu_{i,p_n} = \int f \mathrm{d}\mu_{i,p}.$$

Assumptions

A4 (Continuity of offer set)

The correspondence $p \mapsto \prod_{i \in I} \operatorname{cl} \operatorname{co} X_{i,p}$ is closed at those points such that $\sum_{i=1}^{I} n_i \inf \{p'(x - e_i) \mid x \in X_{i,p}\} = 0$.

- *i.e.*, $p_n \to p$, $x_i^n \in X_{i,p_n}$, and $x_i^n \to x_i^\infty$ implies $x_i^\infty \in X_{i,p}$ for all $i \in I$ whenever $\sum_{i=1}^{I} n_i \inf \{p'(x - e_i) \mid x \in X_{i,p}\} = 0$.
- Vacuous if $\sum_{i=1}^{I} n_i \inf \{p'(x - e_i) \mid x \in X_{i,p}\} < 0$.

Existence

Theorem

Under assumptions A1–A4, a correct expectations equilibrium exists. If A2 is replaced by

A2' *For all p,*

$$\sum_{i=1}^{I} n_i \inf \left\{ p'(x - e_i) \,\middle|\, x \in X_{i,p} \right\} < 0,$$

(in which case A4 is automatic) then all equilibria are non-degenerate.

Outline of the proof

- Define the dual function

$$H_p^*(\xi) = \sum_{i=1}^{I} n_i \left(\xi' e_i + \log \left(\int \mathrm{e}^{-\xi' x} \mathrm{d}\mu_{i,p} \right) \right).$$

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- Box argument. Define

$$\Pi_b(p) = \arg\min_{\xi} \left\{ H_p^*(\xi) \,\middle|\, \xi \geq 0, \|\xi\| \leq b \right\}.$$

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- Normalize $\Phi_b(p) = \{\xi / \|\xi\| \mid \xi \in \Pi_b(p)\}$. Can apply Kakutani to $p \mapsto \Phi_b(p)$. Get "b-quasi equilibrium".

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- Let $b \to \infty$ and get full (degenerate or non-degenerate) equilibrium.

Measure of inefficiency

- Cannot do usual welfare analysis because no utility in model.
- $V_i := \int p'(x - e_i) f_i \mathrm{d}\mu_{i,p}$ is expected ex post average value of transaction for type i.
- The larger V_i is, the more agents are likely to gain from arbitrage.
- Its economy-wide average,

$$A[f;\mu_p] := \sum_{i=1}^{I} n_i V_i = p'(\bar{x}[f;\mu_p] - \bar{e}),$$

is a measure of market inefficiency.

Informational efficiency theorem

Trade-off between efficiency and information gain.

Theorem

If

$$\left(\sum_{i=1}^{I} n_i \operatorname{co} X_{i,p}\right) \cap (-\mathbb{R}^C_{++}) \neq \emptyset,$$

then a feasible and ex ante acceptable allocation $f = (f_i)_{i \in I}$ *is a non-degenerate correct expectations equilibrium distribution if and only if it minimizes the functional*

$$H[g;\mu_p] + tA[g;\mu_p]$$

over unconstrained g *for some* $t \geq 0$*.*

Walrasian eq $\subsetneq$ Bayesian eq

A Walrasian equilibrium is a Bayesian general equilibrium.

Corollary

Let $\mathcal{E} = \{I, \{u_i\}, \{e_i\}\}$ be an endowment economy such that

- $u_i : \mathbb{R}^C_+ \to \mathbb{R}$ *is a continuous, quasi-concave, locally non-satiated utility function of type i agents (with mass $n_i > 0$),*
- *the endowments satisfy $e_i \gg 0$ for all i.*

Then,

1. *there exists an economy $\mathcal{E}'$ such that all Walrasian equilibria of $\mathcal{E}$ are degenerate equilibria of $\mathcal{E}'$,*
2. *the existence of Walrasian equilibria can be shown by using Bayesian general equilibrium theory.*

Outline of the proof

- Take $b > 0$ large enough ($\sum_i n_i e_i \ll b\mathbf{1}$).
  Let the constrained indirect utility be

$$v_i^b(p) := \max\left\{u_i(x) : p'x \leq p'e_i, x \in [0, b]^C\right\}.$$

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Fig

- All assumptions of existence theorem satisfied.
  Can show all Bayesian general equilibria are degenerate.
- By construction Bayesian general equilibria are also Walrasian.

A search model

- A search model in labor market similar to McCall (1970), with a twist of Bayesian general equilibrium.
- Households receive wage offers that they believe to come from a prior P.
- Households can either accept the offer or reject and enjoy unemployment compensation c.
- Assume an endowment economy (*i.e.*, per capita GDP given at y).
- Actual wage distribution determined by MaxEnt.
- Agents update the prior to last period's actual wage distribution.

- Bellman equation:

$$v(w) = \max\left\{\frac{u(w)}{1-\beta}, u(c) + \beta\,\mathrm{E}[v(w')]\right\}.$$

- Reservation wage determined by

$$u(\bar{w}) = u(c) + \frac{\beta}{1-\beta}\int_{\bar{w}}^{\infty}(u(w) - u(\bar{w}))P(\mathrm{d}w).$$

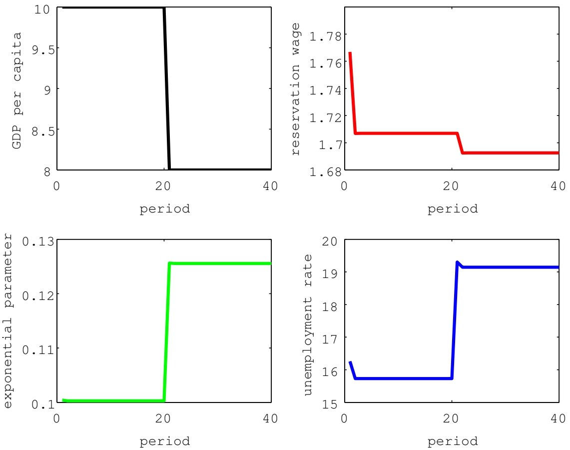
- Dual function:

$$H^*(\pi) = \pi y + \log\left(\mathrm{e}^{-\pi c}P(\bar{w}) + \int_{\bar{w}}^{\infty}\mathrm{e}^{-\pi w}P(\mathrm{d}w)\right).$$

- Entropy price π determined by $(H^*)'(\pi) = 0$.

Numerical example

- Utility function $u(w) = \frac{1}{a}\mathrm{e}^{-aw}$,
- GDP per capita $y_t = \begin{cases} 10, & (t \le 20) \\ 8, & (t > 20) \end{cases}$
  thus a "recession" at $t = 21$.
- $a = 3$, $\beta = 0.9$, $c = 1$,
- Initial prior: uniform on $[0, \infty]$.



- Reservation wage hardly reacts.
- Wage dispersion ↓, unemployment rate ↑

Conclusion

- Propose a general equilibrium theory that replace agent optimization by satisficing behavior.
- Key tool: equivalence between Bayes rule and minimum Kullback-Leibler information (relative entropy) in large samples.
- Prove
    - existence of equilibrium,
    - informational efficiency of equilibrium,
    - Walrasian equilibrium is a special case of Bayesian general equilibrium.

Computation of equilibria

In general, similar to Newton-Raphson method.

- Take initial p_0, π_0.
- Iterate over

$$\pi_{k+1} = \pi_k - [D_\xi^2 H_{p_k}^*(\pi_k)]^{-1} D_\xi H_{p_k}^*(\pi_k),$$
$$p_{k+1} = \pi_{k+1} / \left\| \pi_{k+1} \right\|_1,$$

where

$$H_p^*(\xi) = \sum_{i=1}^{I} n_i \left(\xi' e_i + \log \left(\int \mathrm{e}^{-\xi' x} \mu_{i,p}(\mathrm{d}x) \right) \right).$$

Computation of equilibria

If offer sets are of the form $X_{i,p} = x_{i,p} + \mathbb{R}^C_+$, then reduces to solving

$$\forall c, 1 = -tp_c \sum_{i=1}^{I} n_i x_{ic,p},$$
$$\sum_{c=1}^{C} p_c = 1.$$

These are $C+1$ equations in $C+1$ $(p_1, \dots, p_C, t)$ unknowns.
t: market tightness; $\pi = tp$: entropy price.