Pareto Extrapolation

An Analytical Framework for Studying Tail Inequality

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Background and questions

- Macroeconomic models increasingly incorporate heterogeneity
  - Who gains/loses from policy?
  - How does effectiveness of policy depend on heterogeneity?
- Empirically, wealth distribution has Pareto tail
  - We want to work with such models
  - Actually, people *are* working with such models —without realizing (e.g., Krusell & Smith, 1998)

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  - We want to work with such models
  - Actually, people *are* working with such models —without realizing (e.g., Krusell & Smith, 1998)
- However, solving models with Pareto tails is nontrivial:
  - Analytical methods require strong assumptions
  - Existing numerical methods miss upper tail
- What is lacking: systematic approach for analyzing and solving heterogeneous-agent models that
  1. generate fat-tailed wealth distributions but
  2. do not admit closed-form solutions

This paper

- Propose new systematic approach ("Pareto Extrapolation") for tackling heterogeneous-agent models with fat-tailed wealth distributions numerically
- Our method makes solution algorithm more
  1. transparent
     ($\because$ exploit economic theory as much as possible)
  2. efficient
     ($\because$ narrow down equilibrium object/use good initial guess)
  3. accurate
     ($\because$ correct for truncation error)

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     ($\because$ exploit economic theory as much as possible)
  2. efficient
     ($\because$ narrow down equilibrium object/use good initial guess)
  3. accurate
     ($\because$ correct for truncation error)
- Achieve all of above with zero additional computational cost
- You should use it. MATLAB code available.

Basic idea

- Asymptotic analysis: for rich agents, only multiplicative shocks matter
    1. Analytically solve "asymptotic" problem (e.g., no labor income, no borrowing constraint)
    2. Compute Pareto exponent ζ from random growth theory (Beare & Toda, 2017)
    3. Impose parametric restrictions based on economic theory (e.g., existence of equilibrium, finite mean)
- Pareto extrapolation: use Pareto exponent ζ to correct for truncation error when
    1. constructing transition probability matrix from law of motion
    2. computing aggregate quantities in market clearing condition

Applications

1. Evaluate solution accuracy using analytically solvable Aiyagari model with Pareto tail
   - Pareto extrapolation: $\sim$ 100 times more accurate than existing method
   - Existing method significantly underestimates top wealth shares
2. - Solve a Merton-Bewley-Aiyagari (MBA) model that features
     (a) persistent idiosyncratic endowment and investment risk
     (b) borrowing constraint
     (c) portfolio decision
     (d) recursive utility
     (e) endogenous Pareto-tailed wealth distribution

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     (a) persistent idiosyncratic endowment and investment risk
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     (d) recursive utility
     (e) endogenous Pareto-tailed wealth distribution
   - Introduce 2% wealth tax, with extra tax revenue rebated as consumption subsidy
   - Billionaire wealth tax has unintended consequences
     (a) Little effect on inequality
     (b) Bottom 90% experiences 0.17% welfare loss

Literature

1. Quantitative macro with realistic wealth distribution:
   Krusell & Smith (1998), Quadrini (2000), Cagetti & De Nardi (2006), Krueger, Mitman, & Perri (2016), Carroll, Slacalek, Tokuoka, & White (2017), McKay (2017)
2. Analytical models with Pareto tails:
   Nirei & Souma (2007), Benhabib, Bisin, & Zhu, (2011, 2015, 2016), Moll (2014), Toda (2014, 2019), Toda & Walsh (2015, 2017), Aoki & Nirei (2016, 2017), Gabaix, Lasry, Lions, & Moll (2016), Cao & Luo (2017)
3. Characterization of Pareto exponent:
   Kesten (1973), Reed (2001), Beare & Toda (2017)
4. Solution methods for heterogeneous-agent models:
   Algan, Allais, & Den Haan (2008), Young (2010), Winberry (2018)
5. Bridging theoretical and quantitative analysis:
   Achdou, Han, Lasry, Lions, & Moll (2017)

Issues with existing algorithms

- Basic approach in existing methods ("Truncation")
  1. Set up wealth grid $\mathcal{W}_N = \{w_n\}_{n=1}^N$ and exogenous states $s \in S = \{1, \dots, S\}$; solve individual optimization problem on $S \times N$ grid by dynamic programming
  2. Construct $SN \times SN$ transition probability matrix Q from individual law of motion; compute stationary distribution $Q'\pi = \pi$
  3. Impose market clearing $W = \sum_{s=1}^S \sum_{n=1}^N \pi_{sn} w_n$

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  3. Impose market clearing $W = \sum_{s=1}^S \sum_{n=1}^N \pi_{sn} w_n$
- Issue: largest grid point w_N really represents interval $[w_N, \infty)$
- Hence if wealth distribution is fat-tailed, Truncation method
  1. overestimates transition probability $\Pr(sN \to s'n')$ for $n' < N$; hence underestimates tail probability π_{sN}
  2. underestimates rich agents' wealth $\mathrm{E}\,[w \mid w \geq w_N]$

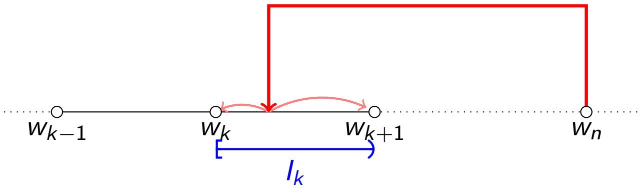
Pareto extrapolation

- For now, assume Pareto exponent ζ known
  (How to compute: later)
- Use Pareto exponent ζ to correct for truncation error:
  1. Imagine hypothetical infinite grid $\mathcal{W}_\infty = \{w_n\}_{n=1}^\infty$, with $w_n = w_N + (n - N)h$ for some $h > 0$ for $n > N$
  2. Set
     $$r_n := (\text{Probability of grid point } n) \propto w_n^{-\zeta-1}$$
  3. Use above to modify law of motion & calculation of integral

Constructing transition probability, $n < N$

- Law of motion: $w' = g_{ss'}(w_n)$
- Suppose $w' \in I_k = [w_k, w_{k+1})$
- Use Young (2010)'s "nonstochastic simulation":
  Assign probabilities to match law of motion on average

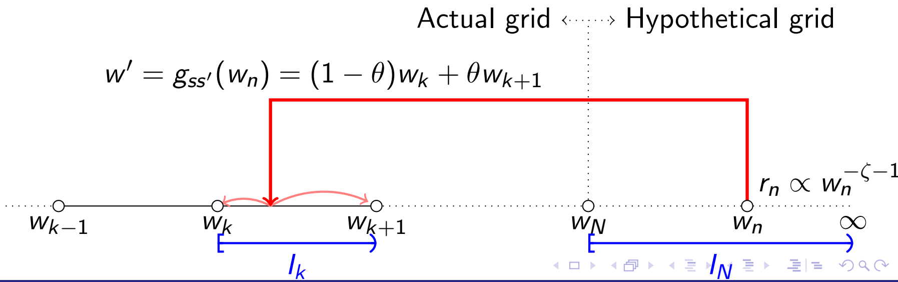
$$w' = g_{ss'}(w_n) = (1-\theta)w_k + \theta w_{k+1}$$



Constructing transition probability, $n = N$

- Hypothetical grid $\mathcal{W}_\infty = \{w_n\}_{n=1}^{\infty}$, with $w_n = w_N + (n-N)h$ for some $h > 0$ for $n > N$; same as before, except $r_n \propto w_n^{-\zeta-1}$
- Zero additional computational cost because correction is only for $n = N$ (single point)

```
[Q,pi] = getQ(PS,PJ,p,x0,xGrid,g,G,zeta)
```



Computing aggregate wealth

- Existing method: $\mathrm{E}[w] = \sum_{s=1}^{S} \sum_{n=1}^{N} \pi_{sn} w_n$
- But w is Pareto-distributed with exponent ζ for $w \in [w_N, \infty)$
- Pareto density $f(x) = \zeta w_N^{\zeta} x^{-\zeta-1}$, so conditional mean is

$$\int_{w_N}^{\infty} x \zeta w_N^{\zeta} x^{-\zeta-1} \, \mathrm{d}x = \frac{\zeta}{\zeta - 1} w_N = w_N + \frac{1}{\zeta - 1} w_N$$

- Thus aggregate wealth approximated by

$$\mathrm{E}[w] = \sum_{s=1}^{S} \left(\sum_{n=1}^{N} \pi_{sn} w_n + \frac{1}{\zeta - 1} \pi_{sN} w_N \right)$$

- If compute $\mathrm{E}[w^{\nu}]$, correction term is $\frac{\nu}{\zeta - \nu}$

How to compute Pareto exponent?

- Remaining issue: get Pareto exponent ζ
- Can compute ζ from
  1. analytical solution to "asymptotic problem" and
  2. Beare & Toda (2017) formula

Asymptotic problem

- For rich agents, labor (additive) income irrelevant
  $\implies$ Focus on capital (multiplicative) income
- Example: income fluctuation problem with stochastic returns

$$\begin{aligned} \text{maximize} \quad & \mathrm{E}_0 \sum_{t=0}^{\infty} [\beta(1-p)]^t \frac{c_t^{1-\gamma}}{1-\gamma} \\ \text{subject to} \quad & w_{t+1} = R_{s_t}(w_t - c_t) + y_{s_{t+1}}, \\ & w_t \ge \underline{w} \end{aligned}$$

- Asymptotic linearity of policy function proved in Ma & Toda (JET 2021)

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- Get closed-form solutions as in Toda (JET 2014, JME 2019) because homogeneous problem

Computing Pareto exponent ζ

- Get asymptotic law of motion $w' = G_{ss'} w$
- Define matrix of conditional moment generating functions by

$$M(z) = \left(\mathrm{E}\left[\mathrm{e}^{z \log G_{ss'}} \,\middle|\, s, s'\right]\right)$$

- Beare & Toda (2017): Pareto exponent $z = \zeta$ solves

$$(1-p)\rho(P \odot M(z)) = 1,$$

where

- ρ: spectral radius (largest eigenvalue)
- P: transition probability matrix
- p: birth/death probability
- $\odot$: Hadamard (entry-wise) product

```
zeta = getZeta(PS,PJ,p,G)
```

Top wealth shares

- If π_N probability assigned to top grid point w_N and Pareto exponent ζ, then tail probability is

$$\Pr(X \geq x) = \int_x^\infty \pi_N \zeta w_N^\zeta x^{-\zeta-1}\,\mathrm{d}x = \pi_N w_N^\zeta x^{-\zeta}$$

- Total wealth held by wealthy agents is

$$\mathrm{E}[X; X \geq x] = \int_x^\infty \pi_N \zeta w_N^\zeta x^{-\zeta}\,\mathrm{d}x = \frac{\zeta}{\zeta-1}\pi_N w_N^\zeta x^{-\zeta+1}$$

- Hence if W aggregate wealth, top p fractile wealth share analytically computed as

$$s(p) = \frac{\zeta}{\zeta-1}\pi_N^{1/\zeta}\frac{w_N}{W}p^{1-1/\zeta}$$

Top tail type and exit probabilities

- Recall Pareto exponent formula $(1-p)\rho(P \odot M(\zeta)) = 1$
- By Perron-Frobenius theorem, $P \odot M(\zeta)$ has left Perron vector $\bar{\pi} \gg 0$ such that $\sum_{s=1}^{S} \bar{\pi}_s = 1$

Proposition

Left Perron vector $\bar{\pi}$ is top tail type distribution:

$$\lim_{w\to\infty} \Pr(s_t = s \mid w_t > w) = \bar{\pi}_s.$$

Furthermore, conditional top tail exit probability given by

$$\lim_{w\to\infty} \Pr(w_{t+1} \le w \mid w_t > w, s_t = s) = 1 - (1-p)\,\mathrm{E}\left[\min\left\{1, G_{t+1}^{\zeta}\right\} \,\middle|\, s_t = s\right].$$

Evaluating solution accuracy

- Consider Krusell-Smith (1998) model, though for analytical tractability assume
  1. No aggregate risk
  2. No income risk
  3. Idiosyncratic discount factor risk and random birth/death only
  4. Subsistence consumption with natural borrowing constraint
- Due to homogeneity, there exists semi-analytical solution
- Compare numerical solutions using
  1. Pareto extrapolation (new method)
  2. Truncation (existing method)

Model: households

- Budget constraint (with natural borrowing limit)

$$a_{t+1} = \frac{R}{1-p}(a_t - c_t + \omega),$$

where R: gross risk-free rate, p: birth/death probability, ω: wage, a_t: financial wealth excluding current income

- Agent maximizes utility

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \left(\prod_{i=0}^{t-1} \beta_{s_i}(1-p) \right) \log(c_t - \underline{c}),$$

  - β_s: discount factor in state $s \in S = \{1, \dots, S\}$ (with transition probability matrix $P = (p_{ss'})$ and stationary distribution $\pi = (\pi_s)$)
  - $\underline{c} = \phi\omega$: subsistence consumption with $\phi \in [0, 1)$

Model: firm

- Representative firm with Cobb-Douglas production function $AK^\alpha L^{1-\alpha} + (1-\delta)K$
- Rents capital and labor from households
- In equilibrium, $L = 1$ and first-order condition

$$R = A\alpha K^{\alpha-1} + 1 - \delta$$

Equilibrium

Stationary equilibrium consists of R, ω, K, optimal consumption rules $\{c_s(a)\}_{s=1}^S$, and stationary distributions $\Gamma = \Gamma(a, s)$ such that

1. R satisfies firm FOC
2. $c_s(a)$ solves utility maximization problem
3. capital market clears, so

$$RK = \sum_{s=1}^{S} \pi_s \int a\Gamma(\mathrm{d}a, s),$$

4. Γ is stationary distribution of law of motion

$$(a, s) \mapsto \begin{cases} \left(\frac{R}{1-p}(a - c_s(a) + \omega), s'\right) & \text{w.p. } (1-p)p_{ss'}, \\ (0, s') & \text{w.p. } p\pi_{s'} \end{cases}$$

Parameter values of analytical model

- Exogenously set parameters:

Description	Symbol	Value
Birth/death probability	p	0.025
Discount factor	β_s	$(0.9, 0.95, 1)$
TFP	A	1
Capital share	α	0.38
Capital depreciation rate	δ	0.08

- Assume

$$P = \begin{bmatrix} 1-q & q & 0 \\ q/2 & 1-q & q/2 \\ 0 & q & 1-q \end{bmatrix}$$

Calibration of q, ϕ

- Calibrate q, ϕ to match wealth distribution
- Get $q = 0.0927$ and $\phi = 0.7286$ with targeted moments:

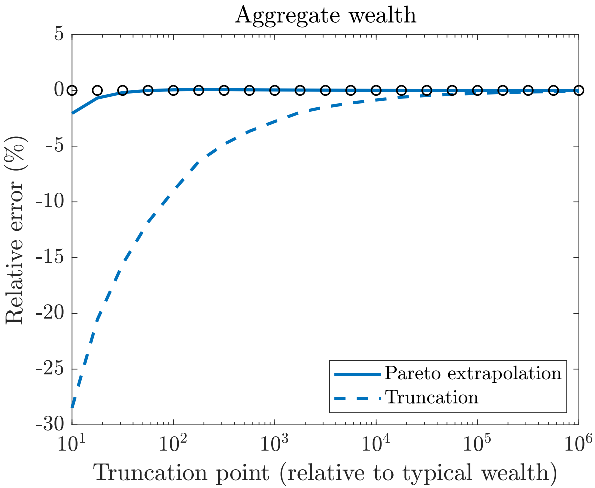
Moment	Data	Model
Top 1% wealth share	32.38	32.90
Top 10% wealth share	69.95	68.72
Top 50% wealth share	98.71	98.11
Pareto exponent	1.52	1.55

Relative error (%) in aggregate capital

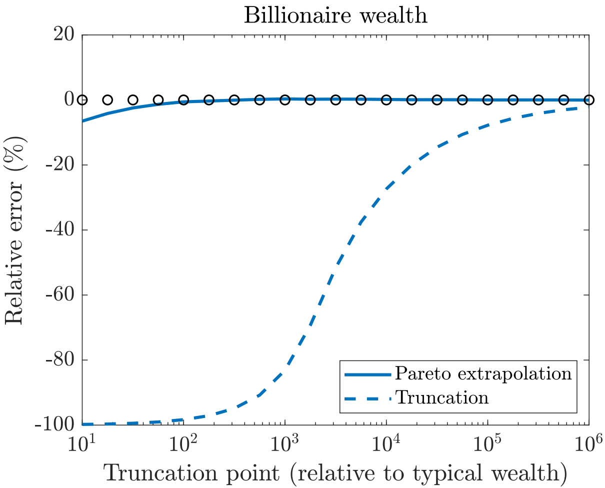
- Compute aggregate capital numerically using true policies
- Use affine-exponential grid details
- N: number of grid points; $\bar{w}$: wealth truncation point

Method:	Truncation			Pareto extrapolation		
$\bar{w}/K_{\mathrm{RA}}$	$N=10$	100	1,000	$N=10$	100	1,000
10^1	-36.920	-26.850	-26.530	-3.195	-1.882	-2.260
10^2	-21.520	-8.690	-7.860	-0.437	0.062	-0.031
10^3	-15.640	-2.950	-2.240	-0.261	0.036	0.017
10^4	-12.930	-1.080	-0.640	-0.234	0.011	0.008
10^5	-11.510	-0.420	-0.180	-0.221	0.003	0.002
10^6	-10.720	-0.180	-0.050	-0.212	0.001	0.001

Relative error in aggregate wealth (%)



Relative error in billionaire wealth (%)



Merton-Bewley-Aiyagari (MBA) model

- Unit continuum of ex ante identical, infinitely lived EZ agents

$$U_t = \left((1-\beta)c_t^{1-1/\varepsilon} + \beta \, \mathrm{E}_t[U_{t+1}^{1-\gamma}]^{\frac{1-1/\varepsilon}{1-\gamma}}\right)^{\frac{1}{1-1/\varepsilon}}$$

- $s \in S = \{1, \dots, S\}$: Markov chain
- Labor productivity: h_s; investment ability: z_s
- Tax on labor, capital income, and estate: τ_h, τ_k, τ_e
- Ex post investment return:

$$\tilde{R}_{sj} = 1 + (1-\tau_k)(R_{sj} - 1),$$
$$\tilde{R}_f = 1 + (1-\tau_k)(R_f - 1),$$

where $R_{sj} = z_s \mathrm{e}^{\epsilon_j} R_f$ and ϵ: IID with values $\epsilon_1, \dots, \epsilon_J$

- Budget constraint

$$w' = \tilde{R}_f(w + (1-\tau_h)\omega h_s - I - c) + \tilde{R}_{sj} I \geq \underline{w}$$

Markov chain

- Agent (head of household) dies with probability p each period, replaced by heir
- Ability state s conditional on survival has transition probability matrix $P_v = (p_{v,ss'})$
- Ability state s conditional on death has transition probability matrix $P_d = (p_{d,ss'})$
- Convenient to define matrix $P(z) := (1-p)P_v + p(1-\tau_e)^z P_d$

Asymptotic problem

Looks complicated, but really EASY...

Proposition

$\rho_s := \max_\theta \mathrm{E}\left[(\tilde{R}_{sj}\theta + \tilde{R}_f(1-\theta))^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}$ *exists. Letting* $D = \operatorname{diag}(\rho_1^{1-\gamma}, \dots, \rho_S^{1-\gamma})$, *asymptotic problem has a solution iff* $\beta\rho(DP)^{\frac{1-1/\varepsilon}{1-\gamma}} < 1$. *Value function of the asymptotic problem is given by* $v_s(w) = b_s w$, *where* $b = (b_1, \dots, b_S) \gg 0$ *is the unique positive solution to the system of nonlinear equations*

$$b_s = \left((1-\beta)^\varepsilon + \beta^\varepsilon \left(\rho_s \left(\sum_{s'=1}^S P_{ss'}(1-\gamma) b_{s'}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}\right)^{\varepsilon-1}\right)^{\frac{1}{\varepsilon-1}}$$

Pareto exponent

Looks complicated, but just apply Beare & Toda (2017) formula

Lemma

Let

$$G_{sj} = (1-(1-\beta)^{\varepsilon} b_s^{1-\varepsilon})(\tilde{R}_{sj}\theta_s^* + \tilde{R}_f(1-\theta_s^*)) > 0$$

be ex post gross growth rate of wealth for asymptotic agents in state (s, j)*. Define conditional MGF of log growth rate by*

$$M_s(z) = \mathrm{E}\left[\mathrm{e}^{z \log G_{sj}} \,\middle|\, s\right] = \sum_{j=1}^{J} p_j G_{sj}^z$$

and $D(z) = \operatorname{diag}(M_1(z), \dots, M_S(z))$*. Suppose* $p_{v,ss} > 0$ *for all* s *and* $G_{sJ} > 1$ *for some* s*. Then there exists a unique solution* $z = \zeta > 1$ *to* $\rho(P(z)D(z)) = 1$*, which is Pareto exponent.*

Parameter values

Parameter	Symbol	Value
Discount factor	β	0.96
Relative risk aversion	γ	2
Elasticity of intertemporal substitution	ε	1
Capital share	α	0.38
Capital depreciation	δ	0.08
Labor income tax	τ_h	0.224
Capital income tax	τ_k	0.25
Estate tax	τ_e	0.4
Borrowing limit	$\underline{w}$	$-\omega/4$

Labor & investment productivity

- $s = (s^\pi, s^\tau)$, where π: permanent; τ: transitory
- s^π takes "low", "high", and "high-entrepreneur"; constant during lifetime of head of household (hence $P_v = I$)
- $h_s = h_s^\pi h_s^\tau$
- $\log h_s^\tau$ is AR(1) calibrated from Guvenen (2009)
- Discretize $\log h_s^\tau$ using Farmer & Toda (QE 2017)
- Calibrate P_d from correlation of income between parent & child
- Set $z_s = z_e$ for entrepreneur, otherwise $z_s = 1$; ϵ is IID normal with variance σ^2, discretize using 3-point Gauss-Hermite

Calibration of z_e, σ

- Calibrate z_e, σ to match wealth distribution
- Get $(z_e, \sigma) = (1.0625, 0.3299)$

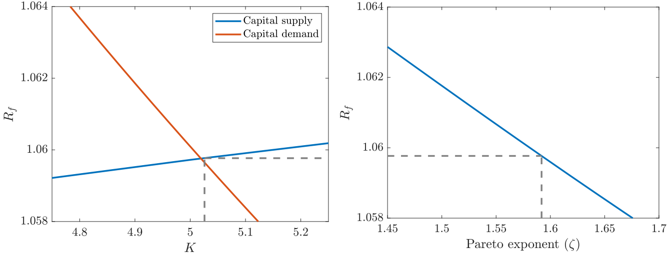
Moment (%)	Data	Model
Top 50% wealth share	99	99
Top 10% wealth share	75	78
Top 1% wealth share	36	42
Top 400 wealth share	2.3	2.4
Top 400 exit rate	5.44	5.63

Validation

- Although not targeted, fraction of entrepreneurs across wealth distribution is close to data

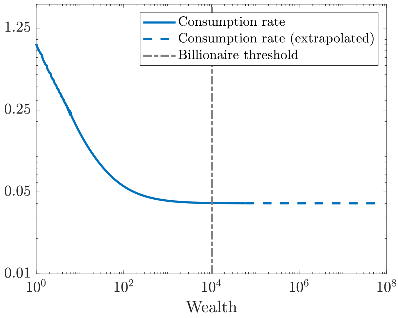
Moment (%)	Data	Model
Share of entrepreneurs in ...		
Top 1%	65	61
Top 5%	51	58
Top 10%	42	48
Top 20%	30	32
Wealth share of entrepreneurs	48	47

Equilibrium

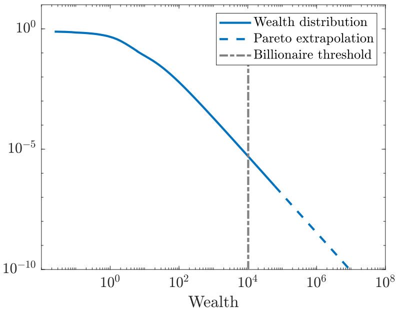


Object	Symbol	Value
Capital-output ratio	K/Y	2.72
Consumption-output ratio	C/Y	0.60
Tax revenue share	T/Y	0.21
Risk-free rate	$R_f - 1$	5.96%
Pareto exponent	ζ	1.59

Extrapolation



(a) Consumption rate.

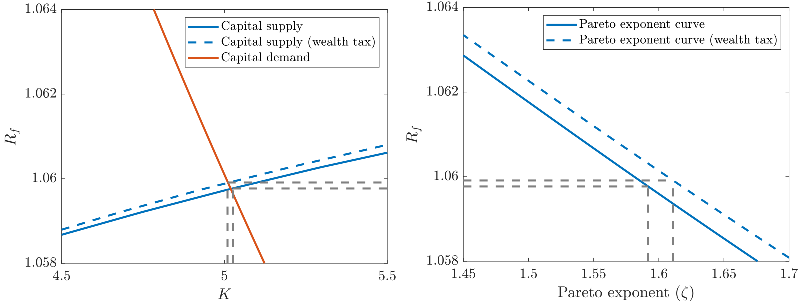


(b) Complementary CDF.

Introducing wealth tax

- Study effect of 2% wealth tax above some threshold
- Threshold is either billionaire (top 0.0005%) or top 1%
- Extra tax revenue rebated to all households as consumption subsidy

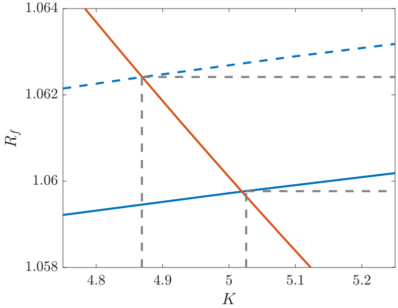
Billionaire wealth tax



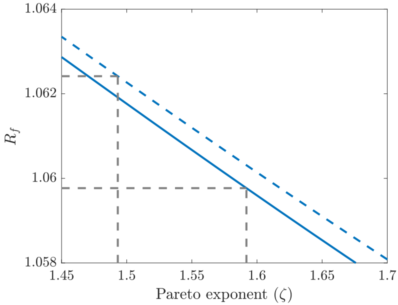
(a) Capital.

(b) Tail inequality.

Top 1% wealth tax



(a) Capital (wealth tax on top 1%).



(b) Tail inequality (wealth tax on top 1%).

Effect of wealth tax on aggregates (%)

Object	Symbol	Wealth Tax on Billionaires	Wealth Tax on Top 1%
Output	Y	-0.06	-1.08
Capital	K	-0.17	-2.82
Risk-free rate	R_f	0.02	0.25
Wage rate	ω	-0.06	-1.08
Tax revenue	T	0.17	3.57

Effect of wealth tax on wealth shares (%)

Object	Wealth Tax on		
	None	Billionaires	Top 1%
Bottom 50%	1.27	1.33	2.36
Next 40%	20.5	20.8	26.0
Next 9%	35.8	36.2	38.6
Top 1% (excl. billionaires)	39.7	40.1	29.5
Billionaires	2.74	1.78	2.84

Effect of wealth tax on welfare (%)

τ_c	0	-0.74%	0	-1.35%
Groups	Billionaire Wealth Tax		Top 1% Wealth Tax	
Bottom 50%	-0.24	-0.17	-0.65	0.69
Next 40%	-0.24	-0.17	1.43	2.80
Next 9%	0.25	0.32	2.08	3.46
Top 1%\ bill.	1.40	1.47	-28.2	-27.2
Billionaires	-23.8	-23.7	-36.5	-35.7
Welfare ($\mathscr{W}$)	-0.21	0.22	-0.15	1.47

Conclusion

- Pareto extrapolation: systematic approach for solving heterogeneous-agent models with Pareto tails
- Pareto (1897):

  > *Nous sommes tout de suite frappé du fait que les points ainsi déterminés, ont une tendance très marqué à se disposer en ligne droite.*
  > *(We are instantly struck by the fact that the points determined this way have a very marked tendency to be disposed in straight line.)*

- Our approach is more transparent, efficient, and accurate at zero additional computational cost

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  (We are instantly struck by the fact that the points determined this way have a very marked tendency to be disposed in straight line.)

- Our approach is more transparent, efficient, and accurate at zero additional computational cost
- No excuse! MATLAB code available
  `https://github.com/alexisakira/Pareto-extrapolation`

Transition probability matrix is sparse

- P is $SN \times SN$, so $(SN)^2$ elements
- For $s, s', n < N$, law of motion takes at most 2 values with positive probability
  $\implies$ at most $2S^2(N-1)$ nonzero elements
- For $s, s', n = N$, law of motion can take N values with positive probability
  $\implies$ at most $S^2 N$ nonzero elements
- Hence fraction of nonzero elements is at most

$$\frac{2S^2(N-1) + S^2 N}{(SN)^2} = \frac{3N-2}{N^2} \to 0$$

as $N \to \infty$, so P is sparse
(zero additional computational cost) ▸ go back

Simulation is bad with Pareto

- Sample mean $\frac{1}{I}\sum_{i=1}^{I} w_i$ converges by LLN, but convergence rate $I^{1-1/\zeta}$ very slow with Pareto exponent $\zeta < 2$
- With $\zeta = 1.1$ and 1% error, $I = 10^{22}$ ($\approx$ # of sand grains) necessary

Sample size I	Pareto exponent ζ			
	≥ 2	1.5	1.3	1.1
$10^0 = 1$	1.00000	1.00000	1.00000	1.00000
10^2	0.10000	0.21544	0.34551	0.65793
10^4	0.01000	0.04642	0.11938	0.43288
10^6	0.00100	0.01000	0.04125	0.28480
10^8	0.00010	0.00215	0.01425	0.18738
10^{10}	0.00001	0.00046	0.00492	0.12328

Simulating Aiyagari model

Simulation is

- very slow (time (sec) is for *one* simulation)
- very inaccurate ▸ go back

T	I	Relative error (%)	Dispersion (%)	Time (sec)
1,000	10^3	21.93	92.97	0.07
	10^4	10.97	47.99	0.38
	10^5	6.70	27.53	3.13
	10^6	6.69	16.33	41.23
10,000	10^3	23.76	96.68	0.39
	10^4	20.42	50.07	3.76
	10^5	9.69	27.47	31.43
	10^6	6.64	16.07	414.30

Constructing exponential grid

- Suppose want to construct N-point exponential grid on (a, b)
- Issue: $a < 0$ possible

1. Choose a shift parameter $s > -a$
2. Construct an N-point evenly-spaced grid on $(\log(a+s), \log(b+s))$
3. Take the exponential
4. Subtract s

- Choose s to cover "typical scale" (e.g., median point corresponds to representative-agent model) go back