Zipf's Law: A Microfoundation

Alexis Akira Toda

University of California San Diego

April 28, 2017

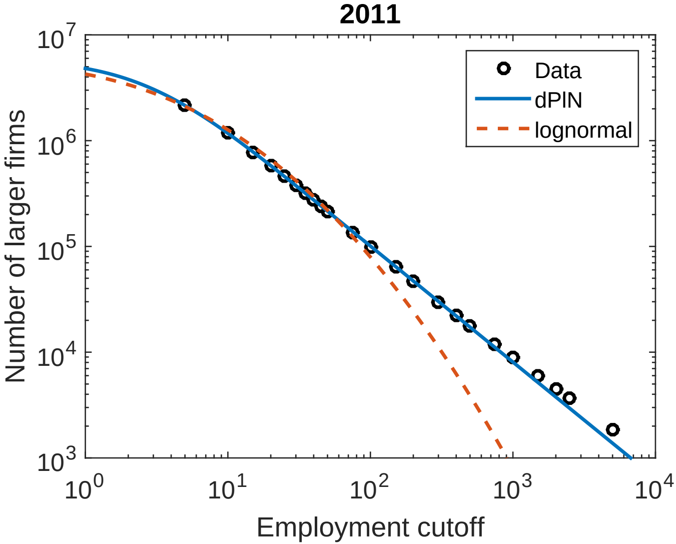
Power laws and Zipf's law

- Size distribution of many economic variables obeys power law:

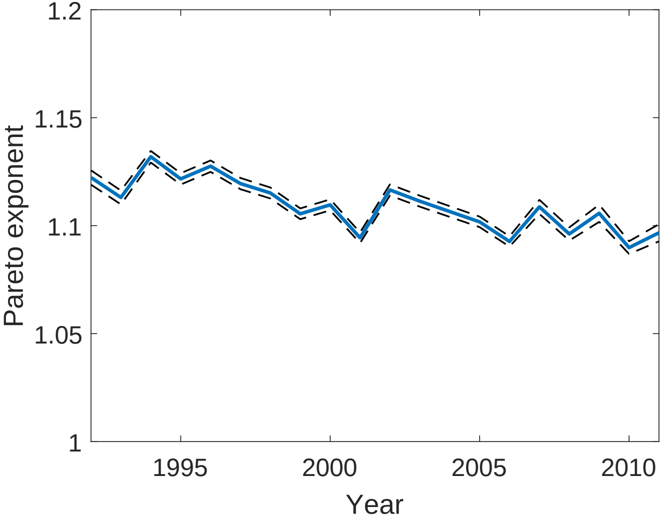
$$P(X > x) \sim x^{-\alpha}$$

  - income & wealth (Pareto, 1896), $\alpha \approx 1.5$-3
  - cities (Auerbach, 1913; Zipf, 1949), $\alpha \approx 1$
  - firms (Axtell, 2001), $\alpha \approx 1$
  - consumption (Toda & Walsh, 2015), $\alpha \approx 4$
  - ...
- Special case: power law with $\alpha \approx 1$ is called Zipf's law and empirically holds for cities and firms

U.S. 2011 Census of firm size



Pareto exponent over time



Existing explanations

- Krugman (1996):

  "There must be a compelling explanation of the astonishing empirical regularity."

Existing explanations

- Krugman (1996):

  "There must be a compelling explanation of the astonishing empirical regularity."

- Most existing explanations use random growth model
  - Simon (1955), Simon & Bonini (1958), Gabaix (1999), Luttmer (2007), Aoki & Nirei (2017), ...

Existing explanations

- Krugman (1996):

  "There must be a compelling explanation of the astonishing empirical regularity."

- Most existing explanations use random growth model
  - Simon (1955), Simon & Bonini (1958), Gabaix (1999), Luttmer (2007), Aoki & Nirei (2017), ...
- However, all require assumptions coming outside of model, such as
  - minimum size
  - small expected growth rate of existing units
- Hence explanation of Zipf's law remains incomplete

Contribution

- I propose a heterogeneous-agent, dynamic general equilibrium model that explains Zipf's law without ad hoc assumptions
- Key ingredients:
  1. Gibrat's law of proportional growth (homothetic preferences, constant-returns-to-scale technology, multiplicative shocks)
  2. Constant probability of birth/death
  3. Production factor in limited supply (new)

Contribution

- I propose a heterogeneous-agent, dynamic general equilibrium model that explains Zipf's law without ad hoc assumptions
- Key ingredients:
  1. Gibrat's law of proportional growth (homothetic preferences, constant-returns-to-scale technology, multiplicative shocks)
  2. Constant probability of birth/death
  3. Production factor in limited supply (new)
- Intuition:
  - Well-known that $1+2$ generates power law (double Pareto distribution (Reed, 2001))
  - With CRS technology and factor in limited supply (labor), decreasing returns at aggregate level (0 aggregate growth)
    $\Longrightarrow$ low growth of individual units in stationary equilibrium
    $\Longrightarrow$ get Zipf's law via endogenous low growth

Contribution

- I propose a heterogeneous-agent, dynamic general equilibrium model that explains Zipf's law without ad hoc assumptions
- Key ingredients:
  1. Gibrat's law of proportional growth (homothetic preferences, constant-returns-to-scale technology, multiplicative shocks)
  2. Constant probability of birth/death
  3. Production factor in limited supply (new)
- Intuition:
  - Well-known that $1+2$ generates power law (double Pareto distribution (Reed, 2001))
  - With CRS technology and factor in limited supply (labor), decreasing returns at aggregate level (0 aggregate growth)
    $\implies$ low growth of individual units in stationary equilibrium
    $\implies$ get Zipf's law via endogenous low growth
- Model consistent with the fact that Zipf's law empirically holds only for cities and firms, which consist of people (inelastic)

Literature

Power law Champernowne (1953), Wold & Whittle (1957), Gabaix (1999, 2009), Nirei & Souma (2007), Benhabib, Bisin, & Zhu (2011, 2015), Arkolakis (2016), Geerolf (2016), Nirei & Aoki (2016)

Zipf's law Auerbach (1913), Zipf (1949), Simon (1955), Simon & Bonini (1958), Rosen & Resnick (1980), Gabaix (1999), Axtell (2001), Luttmer (2007), Aoki & Nirei (2017)

Double power law Reed (2001), Toda (2014), Toda & Walsh (2015), Benhabib, Bisin, & Zhu (2016), Beare & Toda (2016)

Random growth model

- Virtually all existing explanations use random growth model

$$\mathrm{d}X_t = g(X_t)\,\mathrm{d}t + v(X_t)\,\mathrm{d}B_t,$$

  where $g(\cdot)$: drift, $v(\cdot)$: volatility, B_t: Brownian motion
- If $g(x) = gx$ and $v(x) = vx$: geometric Brownian motion (GBM)
- Useful tool to compute cross-sectional distribution: Fokker-Planck equation (Kolmogorov forward equation)

Fokker-Planck equation

- Letting $p(x,t)$ be cross-sectional density, then

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(gp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(v^2 p)$$

- If a stationary density $p(x)$ exists, then $0 = -(gp)' + \frac{1}{2}(v^2 p)''$
- Integrating once, we get $0 = -gp + \frac{1}{2}(v^2 p)'$
- Letting $q = v^2 p$ and solving ODE, we get

$$p(x) = \frac{1}{v(x)^2} \exp\left(\int \frac{2g(x)}{v(x)^2}\,\mathrm{d}x\right),$$

where constant of integration determined by $\int p(x)\,\mathrm{d}x = 1$

Existing explanations

- Consider GBM, so $g(x) = gx$ and $v(x) = vx$
- Assume there is a minimum size $x_{\min} > 0$ and $g < 0$, so that a steady state exists
- Using previous formula, we can show $p(x) = \zeta x_{\min}^{\zeta} x^{-\zeta-1}$ and

$$P(X > x) = \left(\frac{x}{x_{\min}}\right)^{-\zeta},$$

  where $\zeta = 1 - \frac{2g}{v^2} > 1$: Pareto distribution
- If $|g| \ll v^2$ (low growth), then $\zeta \approx 1$: Zipf's law
- Alternatively, since mean size is

$$\bar{x} = \int_{x_{\min}}^{\infty} x p(x) \, \mathrm{d}x = \frac{\zeta}{\zeta - 1} x_{\min},$$

  we get $\zeta = \frac{1}{1 - x_{\min}/\bar{x}}$. Hence $\zeta \approx 1$ if $x_{\min} \ll \bar{x}$ (small minimum size)

Difficulties

1. Existence of a minimum size $x_{\min}$ is an ad hoc assumption
   - In the presence of a minimum size, rational agents will behave differently near and far from the boundary
     $\implies$ will not get GBM in general

Difficulties

1. Existence of a minimum size x_{min} is an ad hoc assumption
   - In the presence of a minimum size, rational agents will behave differently near and far from the boundary
     $\implies$ will not get GBM in general
2. In a fully specified economic model, g, v are endogenous variables, and in general there is no reason to expect that low growth condition $|g| \ll v^2$ holds

GBM with constant birth/death

- Consider GBM with no minimum size, but with constant birth/death at Poisson rate η and initial size x_0
- Fokker-Planck equation in steady state is FPE without death

$$0 = -(gxp)' + \frac{1}{2}(v^2x^2p)'' - \eta p$$

- By solving second order ODE, we can show

$$p(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} x_0^{\alpha} x^{-\alpha-1}, & (x \geq x_0) \\ \frac{\alpha\beta}{\alpha+\beta} x_0^{-\beta} x^{\beta-1}, & (0 < x < x_0) \end{cases}$$

(double Pareto) where $\alpha > 0 > -\beta$ are solutions to quadratic equation

$$\frac{v^2}{2}\zeta^2 + \left(g - \frac{v^2}{2}\right)\zeta - \eta = 0$$

GBM with constant birth/death

- Solving quadratic equation

$$\frac{v^2}{2}\zeta^2 + \left(g - \frac{v^2}{2}\right)\zeta - \eta = 0,$$

upper tail Pareto exponent is

$$\alpha = \frac{1}{2}\left(\sqrt{\left(1 - \frac{2g}{v^2}\right)^2 + \frac{8\eta}{v^2}} + \left(1 - \frac{2g}{v^2}\right)\right)$$

- Hence minimum size is no longer necessary, but to get Zipf's law we still need low growth condition $|g|, \eta \ll v^2$

Minimal model of city (village) size

- Continuum of villages and households, with mass 1 and N
- Single consumption good (potato)
- Village authority hires labor and uses stock of potatoes to produce new potatoes
- Villages are hit by idiosyncratic productivity shocks as well as famines (rare disasters)
- When a famine hits, all potatoes wiped out, but village authority receives fraction κ of potatoes of all other villages according to mutual insurance agreement
- Households eat potatoes and migrate across villages freely

Evolution of potatoes

- Letting x_t be stock of potatoes in typical village, dynamics is

$$\mathrm{d}x_t = (F(x_t, n_t) - \omega n_t)\,\mathrm{d}t - \eta\kappa x_t\,\mathrm{d}t + v x_t\,\mathrm{d}B_t,$$

  where n_t: labor input, F: CRS production function, ω: wage, η: Poisson rate of famine, v: idiosyncratic volatility
- Village authority maximizes profits, so solves $n_t = \arg\max_n [F(x_t, n) - \omega n]$
- Letting $f(x) = F(x, 1)$, by FOC $\omega = f(y) - yf'(y)$, where $y = x_t/n_t$ is steady state potato-labor ratio
- Substituting into equation of motion,

$$\mathrm{d}x_t = (\mu - \eta\kappa)x_t\,\mathrm{d}t + v x_t\,\mathrm{d}B_t,$$

  where $\mu = f'(y) \implies$ GBM

Equilibrium condition

- Individual village: $\mathrm{d}x_t = (\mu - \eta\kappa)x_t \,\mathrm{d}t + vx_t \,\mathrm{d}B_t$
- In a short time interval Δt, $\eta\Delta t$ villages experience famine, so aggregate stock of potatoes X satisfies

$$\begin{aligned} X + \Delta X = & \underbrace{(1-\eta\Delta t)(1+(\mu-\eta\kappa)\Delta t)X}_{\text{Aggregate potatoes of non-famine villages}} \\ & + \underbrace{(\eta\Delta t)(\kappa X)}_{\text{Aggregate potatoes of famine villages}} \\ =& (1+(\mu-\eta)\Delta t)X + \text{higher order terms.} \end{aligned}$$

- Subtracting X from both sides and letting $\Delta t \to 0$, we obtain $\mathrm{d}X = (\mu - \eta)X \,\mathrm{d}t$
- In steady state, $X =$ constant, so it must be $\mu = \eta$: endogenous low growth $\mathrm{d}x_t = \eta(1-\kappa)x_t \,\mathrm{d}t + vx_t \,\mathrm{d}B_t$

Simple explanation of Zipf's law

Proposition (Zipf's law)

The stationary city size distribution is double Pareto. The upper tail Pareto exponent ζ *satisfies*

$$1 < \zeta < 1 + \frac{2\eta\kappa}{v^2}.$$

As $\eta \to 0$*, we obtain Zipf's law* $\zeta \to 1$*.*

Simple explanation of Zipf's law

Proposition (Zipf's law)

The stationary city size distribution is double Pareto. The upper tail Pareto exponent ζ satisfies

$$1 < \zeta < 1 + \frac{2\eta\kappa}{v^2}.$$

As $\eta \to 0$, we obtain Zipf's law $\zeta \to 1$.

Proof.

- ζ is positive root of $q(\zeta) = \frac{v^2}{2}\zeta^2 + \left(g - \frac{v^2}{2}\right)\zeta - \eta$ ▸ Why? with $g = \eta(1 - \kappa)$
- Can show $q(1) < 0$ and $q(1 + 2\eta\kappa/v^2) > 0$ by direct substitution, so $1 < \zeta < 1 + \frac{2\eta\kappa}{v^2}$ □

General theory

- Above is a minimal model, but Zipf's law holds in a wide variety of GE models (details in paper)
- Ingredients:
  1. an agent type solves a homogeneous problem (homothetic preferences, CRS technology, proportional constraints),
  2. agents enter/exit the economy at small rate $\eta > 0$, and
  3. at least one production factor is in limited supply
- Robust to
  1. elastic labor supply,
  2. balanced growth,
  3. coexistence of Zipf and non-Zipf distributions,
  4. random initial size,
  5. multiple agent types,
  6. discrete time model with non-Gaussian shocks

Model of firm size

- So far, very little optimizing behavior by agents (in order to illustrate the essential mechanism)
- Consider a dynamic general equilibrium model of firm size with fully optimizing agents (entrepreneurs with mass 1 and workers with mass N)
- Workers supply labor, consume, and save/borrow at equilibrium risk-free rate
- Entrepreneurs are born with 1 unit of capital, hire labor, operate CRS technology using capital with idiosyncratic investment risk, and go bankrupt at constant Poisson rate η

Workers

- Workers are infinitely lived and maximize CRRA utility

$$U_t = \int_0^\infty \mathrm{e}^{-\rho s} \frac{c_{t+s}^{1-1/\varepsilon}}{1-1/\varepsilon} \,\mathrm{d}s$$

  subject to budget constraint $\mathrm{d}x_t = (rx_t + \omega - c_t)\,\mathrm{d}t$, where ρ: discount rate, ε: EIS, x_t: financial wealth, r: (equilibrium) risk-free rate, ω: wage
- Letting $w_t = x_t + \omega/r$ be effective wealth, get $\mathrm{d}w_t = (rw_t - c_t)\,\mathrm{d}t$
- Merton (1971)-type optimal consumption-saving problem: solution is $c_t = (\rho\varepsilon + (1-\varepsilon)r)w_t$
- By budget constraint, wealth dynamics is $\mathrm{d}w_t = \varepsilon(r-\rho)w_t\,\mathrm{d}t$

Entrepreneurs

- Epstein-Zin preferences with RRA γ and EIS ε
- Budget constraint is

$$\mathrm{d}x_t = (F(k_t, n_t) - \omega n_t + (r+\eta)b_t - c_t)\,\mathrm{d}t + \sigma k_t\,\mathrm{d}B_t,$$

where k_t: capital, b_t: bond holdings, $x_t = k_t + b_t$: net worth, n_t: labor input, c_t: consumption, ω: wage, η: bankruptcy rate, σ: idiosyncratic volatility

- Letting $y = k_t/n_t$ be equilibrium capital-labor ratio and $\mu = f'(y)$, by same argument as in village economy we get

$$\mathrm{d}x_t = (r_e + (\mu - r_e)\theta - m)x_t\,\mathrm{d}t + \sigma\theta x_t\,\mathrm{d}B_t,$$

where $r_e = r + \eta$: effective risk-free rate, $\theta = k_t/x_t$: leverage, $m = c_t/x_t$: marginal propensity to consume

Entrepreneurs

- Again, standard Merton (1971)-type optimal consumption-saving-portfolio problem. Optimal rules are

$$\theta = \frac{\mu - r_e}{\gamma\sigma^2},$$

$$m = (\rho + \eta)\varepsilon + (1 - \varepsilon)\left(r_e + \frac{(\mu - r_e)^2}{2\gamma\sigma^2}\right).$$

- Substituting into budget constraint, wealth dynamics is

$$\mathrm{d}x_t = gx_t\,\mathrm{d}t + vx_t\,\mathrm{d}B_t$$

(GBM), where drift g and volatility v are given by

$$g = (r - \rho)\varepsilon + (1 + \varepsilon)\frac{(\mu - r_e)^2}{2\gamma\sigma^2},$$

$$v = \sigma\theta = \frac{\mu - r_e}{\gamma\sigma}.$$

Equilibrium

- There is no reason why preference parameters should be the same for workers and entrepreneurs
- Let discount factor and EIS of workers be ρ_W, ε_W, and those for entrepreneurs ρ, ε

Theorem (Existence)

Suppose that $f(x) = F(x, 1)$ satisfies the usual Inada conditions $f' > 0$, $f'' < 0$, $f'(0) = \infty$, and $f'(\infty) \leq 0$. Then a stationary equilibrium exists if and only if

$$\left(1 - \frac{1}{\bar{y}N}\right)\eta > -\rho\varepsilon,$$

where $\bar{y} > 0$ is the (unique) number such that $f'(\bar{y}) = \eta$.
In particular, an equilibrium exists if $\eta > 0$ is sufficiently small.

Zipf's law

- Unlike in the previous village economy, $\kappa = 1/K$ and volatility v are endogenous (K: aggregate capital)
- Hence even though equation of motion

$$\mathrm{d}x_t = \eta(1-\kappa)x_t\,\mathrm{d}t + vx_t\,\mathrm{d}B_t$$

is same as before, $\zeta \to 1$ as $\eta \to 0$ is not obvious

Zipf's law

- Unlike in the previous village economy, $\kappa = 1/K$ and volatility v are endogenous (K: aggregate capital)
- Hence even though equation of motion

$$\mathrm{d}x_t = \eta(1-\kappa)x_t\,\mathrm{d}t + vx_t\,\mathrm{d}B_t$$

is same as before, $\zeta \to 1$ as $\eta \to 0$ is not obvious

Theorem (Zipf's law)

The upper tail Pareto exponent ζ satisfies

$$1 < \zeta < 1 + \frac{2\eta\kappa}{v^2}.$$

As $\eta \to 0$, $\kappa = \frac{1}{K} = \frac{1}{yN}$ is bounded above and $v = \sigma\theta = \frac{f'(y)-r-\eta}{\gamma\sigma}$ is bounded away from 0, so Zipf's law $\zeta \to 1$ holds.

Calibration

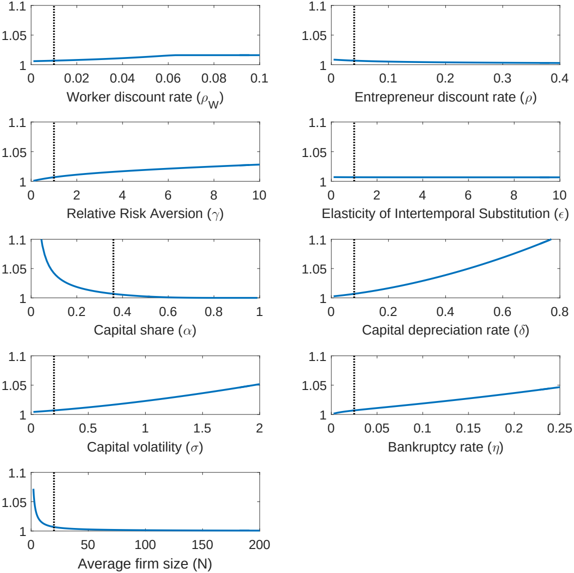
- Cobb-Douglas technology $F(k,n) = k^\alpha n^{1-\alpha} - \delta k$
- Model is completely specified by the parameters $(\rho_W, \rho, \gamma, \varepsilon, \alpha, \delta, \sigma, \eta, N)$
- Preference and technology parameters are $\rho = 0.04$, $\varepsilon = 1$, $\alpha = 0.36$, $\delta = 0.08$, and $\sigma = 0.2$, as in Angeletos (2007)
- $\rho_W = 0.01$ to match historical risk-free rate
- In 2011 U.S. data, 5,684,424 firms employed 113,425,965 workers (19.95 workers per firm on average), so $N = 20$
- $\gamma = 1$ because entrepreneurs should not be so risk averse (also consider $\gamma = 0.5, 2$ for robustness)
- η is bankruptcy rate (2.5% in data, Luttmer (2010)) as well as spread of corporate bonds (about 2%, Gilchrist *et al.* (2009)), so $\eta = 0.025$ (also consider $\eta = 0.05, 0.1$)

Results

Quantity	Symbol	Values				
Risk aversion	γ	1	0.5	2	1	1
Bankruptcy rate (%)	η	2.5	2.5	2.5	5	10
Capital-labor ratio	y	3.49	4.01	2.93	2.58	1.65
Wage	ω	1.004	1.055	0.942	0.900	0.767
Private premium (%)	$\mu - r_e$	4.68	3.31	6.61	5.62	7.13
Equity premium (%)	$\mu - r$	7.18	5.81	9.11	10.62	17.13
Leverage	θ	1.17	1.65	0.83	1.41	1.78
Volatility (%)	v	23.4	33.1	16.5	28.1	35.6
Pareto exponent	ζ	1.007	1.004	1.011	1.011	1.019

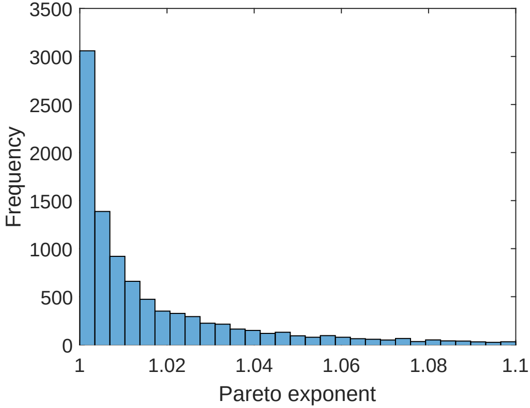
- Riskier environment ($\gamma \uparrow$ or $\eta \uparrow$) leads to high equity premium, low capital, and low wage, but different mechanism
- With $\gamma \uparrow$, risk aversion leads to less leverage and lower volatility (portfolio effect)
- With $\eta \uparrow$, more destruction of capital but higher leverage and volatility due to cheap labor (resource effect)

Sensitivity analysis



Sensitivity analysis

- Randomly generate 10^4 parameters, up to 5-fold change from baseline (for α, uniformly generate from $[0.1\alpha, 1.9\alpha]$)
- mean $= 1.0312$, median $= 1.0089$, 95 percentile $= 1.1313$



Conclusion

- Developed a fully specified, dynamic general equilibrium model that explains Zipf's law (Pareto exponent very close to 1)
- Key ingredients are
  1. Gibrat's law of proportional growth (homothetic preferences, constant-returns-to-scale technology, multiplicative shocks)
  2. Constant probability of birth/death
  3. Production factor in exogenously bounded supply
- Model explains why Zipf's law is observed for cities and firms, which consist of people (inelastic supply)