Geometrically Stopped Markovian Random Growth Processes and Pareto Tails

Brendan K. Beare[1] Alexis Akira Toda[2]

[1]University of Sydney

[2]University of California San Diego

March 10, 2020

This paper

- We study the tail behavior of

$$W_T = \sum_{t=1}^{T} X_t,$$

where

  - $\{X_t\}_{t=1}^{\infty}$: some stochastic process,
  - T: some stopping time.

- Main result: W_T has exponential tails under fairly mild conditions; simple formula for the tail exponent α.
- Example: if $\{X_t\}_{t=1}^{\infty}$ is iid and T is geometric with mean $1/p$, then

$$(1-p)\,\mathrm{E}[\mathrm{e}^{\alpha X}] = 1.$$

Why this problem is interesting

- Many empirical size distributions obey power laws (e.g., city size, firm size, income, consumption, wealth, etc.)

$$\mathrm{P}(S > s) \sim s^{-\alpha},$$

where S: size.

- Popular explanation is "random growth model": $S_t = G_t S_{t-1}$, where G: gross growth rate.
- Taking logarithm and setting $W_t = \log S_t$, $X_t = \log G_t$, we obtain the random walk

$$W_t = W_{t-1} + X_t.$$

Hence if $W_0 = 0$, we have $W_T = \sum_{t=1}^T X_t$.

Questions

- Most existing explanations using random growth assume iid Gaussian environment (geometric Brownian motion; Reed, 2001).
- Given ubiquity of power law distributions in empirical data (likely non-iid and non-Gaussian), generative mechanism should be robust (not depend on iid Gaussian assumptions).

Questions:

1. Do non-Gaussian, Markovian random growth processes generate Pareto tails?
2. If so, how is Pareto exponent determined and how does it depend on exogenous parameters?

Contribution

- Characterize tail behavior of random growth models with non-Gaussian, Markovian shocks.
  1. Analytical determination of Pareto exponent.
  2. Comparative statics.
- Two economic applications:
  1. Characterize Pareto tail behavior of wealth distribution in heterogeneous-agent models with idiosyncratic investment risk.
  2. Estimate random growth model using Japanese municipality population data. Model consistent with observed Pareto exponent but *only after* allowing for Markovian dynamics.

Literature

Power law Wold & Whittle (1957), Simon & Bonini (1958), Gabaix (1999), Reed (2001), Toda (2014)

Tauberian theorem Graham & Vaaler (1981), Nakagawa (2007)

Wealth distribution Benhabib et al. (2011, 2015, 2016), Toda (2014, 2019), Acemoglu & Cao (2015), Toda & Walsh (2015, 2017), Gabaix et al. (2016), Nirei & Aoki (2016), Aoki & Nirei (2017), Cao & Luo (2017), Stachurski & Toda (2019), Ma et al. (2020)

Basic setup

Object of interest:

- We seek to characterize the behavior of tail probabilities

$$\mathrm{P}(W_T > w) \quad \text{and} \quad \mathrm{P}(W_T < -w)$$

as $w \to \infty$, where...

Basic setup

Object of interest:

- We seek to characterize the behavior of tail probabilities

$$\mathrm{P}(W_T > w) \quad \text{and} \quad \mathrm{P}(W_T < -w)$$

as $w \to \infty$, where...

Markov additive process:

- $\{W_t, J_t\}_{t=0}^{\infty}$ is a Markov additive process, which means...

Basic setup

Object of interest:

- We seek to characterize the behavior of tail probabilities

$$\mathrm{P}(W_T > w) \quad \text{and} \quad \mathrm{P}(W_T < -w)$$

as $w \to \infty$, where...

Markov additive process:

- $\{W_t, J_t\}_{t=0}^{\infty}$ is a Markov additive process, which means...

Hidden Markov state:

- $\{J_t\}_{t=0}^{\infty}$ is a time homogeneous Markov chain taking values in $\mathcal{N} = \{1, \dots, N\}$.
- The transition probability matrix is $\Pi = (\pi_{nn'})$, where $\pi_{nn'} = \mathrm{P}(J_1 = n' \mid J_0 = n)$.
- Initial condition: ϖ is the $N \times 1$ vector of probabilities $\mathrm{P}(J_0 = n)$, $n = 1, \dots, N$.

Basic setup

Increment process:

- $W_0 = 0, \ W_t = \sum_{s=1}^{t} X_s$.
- Distribution of increment $X_t = W_t - W_{t-1}$ depends only on $(J_{t-1}, J_t) = (n, n')$. ▸ Markov additive process
- Special cases:
  1. If $N = 1$, then $\{X_t\}_{t=1}^{\infty}$ is iid.
  2. If $X_t =$ constant conditional on J_t, then $\{X_t\}_{t=1}^{\infty}$ is a finite-state Markov chain.

Stopping time:

- $\{W_t\}_{t=0}^{\infty}$ stops with state-dependent probability.
- $v_{nn'} = \mathrm{P}(T > t \mid J_{t-1} = n, J_t = n', T \geq t)$: conditional survival probability.
- $V = (v_{nn'})$: survival probability matrix.

Basic setup

Conditional moment generating function:

- For $s \in \mathbb{R}$, define
$M_{nn'}(s) = \mathrm{E}\left[\mathrm{e}^{sX_1} \,\middle|\, J_0 = n, J_1 = n'\right] \in (0, \infty]$.
- $M(s) = (M_{nn'}(s))$: $N \times N$ matrix of conditional MGFs.

Region of convergence:

- We define

$$\mathcal{I} = \left\{s \in \mathbb{R} : M_{nn'}(s) < \infty \text{ for all } n, n' \in \mathcal{N}\right\}.$$

- $\mathcal{I}$ is an interval containing zero, with possibly infinite endpoints.
- $\mathcal{I}$ is the intersection of the N^2 regions of convergence of the conditional moment generating functions of X_t given $(J_{t-1}, J_t) = (n, n')$.

Assumption

Assumption

1. *The matrix* $V \odot \Pi$ *is irreducible.*
2. *There exists a pair* (n, n') *such that* $v_{nn'} < 1$ *and* $\pi_{nn'} > 0$.

Assumption

Assumption

1. *The matrix $V \odot \Pi$ is irreducible.*
2. *There exists a pair (n, n') such that $v_{nn'} < 1$ and $\pi_{nn'} > 0$.*

- $V \odot \Pi$ is Hadamard (entry-wise) product.
- A matrix A is irreducible if for any pair (n, n'), there exists k such that $|A|^k_{nn'} > 0$.
- Intuitively, irreducibility of $V \odot \Pi$ means we can transition from n to n' eventually without stopping.
- $v_{nn'} < 1$ and $\pi_{nn'} > 0$ guarantees $T < \infty$ almost surely.
- $\rho(A)$: spectral radius (largest absolute value of all eigenvalues) of A.

Main result

Theorem

As a function of $s \in \mathcal{I}$, the spectral radius $\rho(V \odot \Pi \odot M(s))$ is convex and less than 1 at $s = 0$. There can be at most one positive $\alpha \in \mathcal{I}$ such that

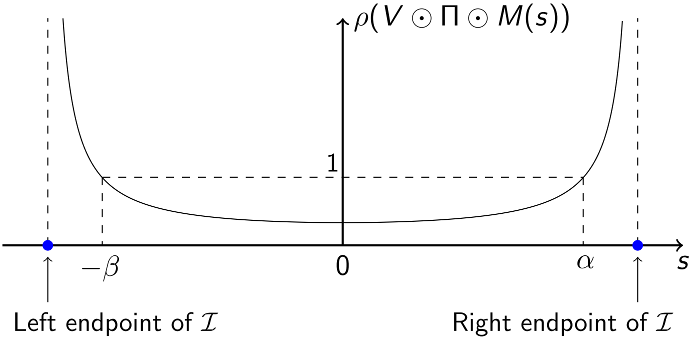
$$\rho(V \odot \Pi \odot M(\alpha)) = 1,$$

and if such α exists in the interior of $\mathcal{I}$ then

$$\lim_{w \to \infty} \frac{1}{w} \log \mathrm{P}(W_T > w) = -\alpha.$$

- Similar statement holds for lower tail ($-\beta < 0$ instead of $\alpha > 0$).

Determination of α and β



Refinement

Theorem

Let everything be as above. Then there exist $A, B > 0$ such that

$$\lim_{w\to\infty} \mathrm{e}^{\alpha w}\mathrm{P}(W_T > w) = A,$$
$$\lim_{w\to\infty} \mathrm{e}^{\beta w}\mathrm{P}(W_T < -w) = B$$

except when there exist $c > 0$ and $a_{nn'} \in \mathbb{R}$ such that

$$\mathrm{supp}(X_1 | J_0 = n, J_1 = n') \subset a_{nn'} + c\mathbb{Z}$$

for all $n, n' \in \mathcal{N}$. (We can take $a_{nn} = 0$ if $v_{nn}\pi_{nn} > 0$.)

Geometrically stopped random growth processes

Theorem

Let everything be as above. Let $S_0 > 0$ *be a random variable independent of* W_T *satisfying* $\mathrm{E}[S_0^{\alpha+\epsilon}] < \infty$ *for some* $\epsilon > 0$*, and define the random variable* $S = S_0 \mathrm{e}^{W_T}$*. Then there exist numbers* $0 < A_1 \leq A_2 < \infty$ *such that*

$$A_1 = \liminf_{s \to \infty} s^\alpha \mathrm{P}(S > s) \leq \limsup_{s \to \infty} s^\alpha \mathrm{P}(S > s) = A_2,$$

with $A_1 = A_2 = A$ *unless unless there exist* $c > 0$ *and* $a_{nn'} \in \mathbb{R}$ *such that* $\mathrm{supp}(X_1 | J_0 = n, J_1 = n') \subset a_{nn'} + c\mathbb{Z}$ *for all* $n, n' \in \mathcal{N}$*.*

- S has a Pareto upper tail with exponent α.

Comparative statics

- Pareto exponent α is implicitly determined by

$$\rho(V \odot \Pi \odot M(\alpha)) = 1.$$

  How does it vary with exogenous parameters?
- Perturb lifespan, growth, volatility, and persistence. Assume
  - distribution of X_t conditional on $(J_{t-1}, J_t) = (n, n')$ is parametrized as $\mu_{nn'} + \sigma_{nn'} Y_{nn'}$, where $Y_{nn'}$ has mean zero,
  - transition probability matrix is $\Pi(\tau) = \tau I + (1-\tau)\Pi$ for some fixed Π.
- Use implicit function theorem.

Comparative statics

Theorem

Let $\theta = (V, \mu, \sigma, \tau)$ be parameters, and suppose assumptions of Theorem satisfied at θ^0. Then there exists an open neighborhood Θ of θ^0 such that Pareto exponent $\alpha(\theta)$ is continuously differentiable on Θ. Furthermore,

1. $\partial\alpha/\partial v_{nn'} \leq 0$: *lifespan* $\uparrow \implies$ *inequality* $\uparrow$.
2. $\partial\alpha/\partial\mu_{nn'} \leq 0$: *growth* $\uparrow \implies$ *inequality* $\uparrow$.
3. $\partial\alpha/\partial\sigma_{nn'} \leq 0$: *volatility* $\uparrow \implies$ *inequality* $\uparrow$.
4. *If* $v_{nn'}$ *and* X_t *depend only on current state* $J_t = n'$, *then* $\partial\alpha/\partial\tau \leq 0$: *persistence* $\uparrow \implies$ *inequality* $\uparrow$.

Proof of main result

- The proof uses several mathematical results:
  1. Nakagawa's Tauberian Theorem and its refinement
  2. Convex inequalities for spectral radius
  3. Perron-Frobenius Theorem
  4. Residue formula for matrix pencil inverses
- For the iid case, we can avoid 2–4 above.

Skip proof

Laplace transform

- For a random variable X with cdf F, let

$$M(s) = \mathrm{E}[\mathrm{e}^{sX}] = \int_{-\infty}^{\infty} \mathrm{e}^{sx}\,\mathrm{d}F(x)$$

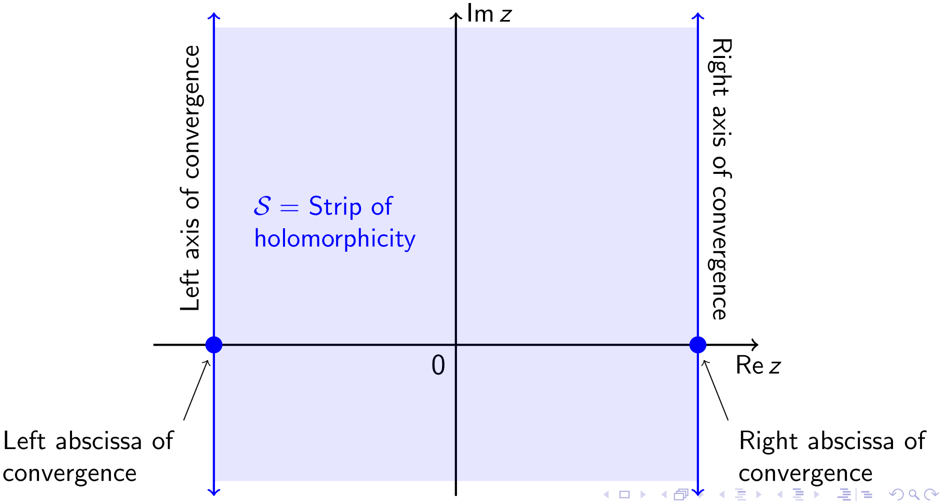
be its moment generating function (mgf), which is also known as the (two-sided) Laplace transform.

- Since e^{sx} convex in s, so is $M(s)$; hence its domain $\mathcal{I} = \{s \in \mathbb{R} : M(s) < \infty\}$ is an interval. Let $-\beta \leq 0 \leq \alpha$ be boundary points (may be 0 or $\pm\infty$).
- For $z \in \mathbb{C}$, by definition of Lebesgue integral,

$$M(z) = \mathrm{E}[\mathrm{e}^{zX}] = \int_{-\infty}^{\infty} \mathrm{e}^{zx}\,\mathrm{d}F(x)$$

exists and finite if and only if $\operatorname{Re} z \in \mathcal{I}$. $M(z)$ holomorphic on strip of analiticity $\mathcal{S} = \{z \in \mathbb{C} : -\beta < \operatorname{Re} z < \alpha\}$.

Strip of holomorphicity



Tauberian theorem

Theorem (Essentially, Theorem 5* of Nakagawa, 2007)

Let X be a real random variable and $M(z) = \mathrm{E}[\mathrm{e}^{zX}]$ its Laplace transform with right abscissa of convergence $0 < \alpha < \infty$ and strip of holomorphicity $\mathcal{S}$. Suppose $A := \lim_{s \uparrow \alpha} (\alpha - s) M(s)$ exists, and let B be the supremum of all $b > 0$ such that $M(z) + A(z - \alpha)^{-1}$ continuously extends to $\mathcal{S}_b^+ = \mathcal{S} \cup \{z \in \mathbb{C} : z = \alpha + it, |t| < b\}$. Suppose that $B > 0$. Then we have

$$\frac{2\pi A/B}{\mathrm{e}^{2\pi\alpha/B} - 1} \leq \liminf_{x \to \infty} \mathrm{e}^{\alpha x} \mathrm{P}(X > x) \leq \limsup_{x \to \infty} \mathrm{e}^{\alpha x} \mathrm{P}(X > x) \leq \frac{2\pi A/B}{1 - \mathrm{e}^{-2\pi\alpha/B}},$$

where the bounds should be read as A/α if $B = \infty$.

Discussion

- By previous result, taking logarithm and letting $x \to \infty$, we get
$$\lim_{x\to\infty} \frac{\log \mathrm{P}(X > x)}{x} = -\alpha,$$
which is Nakagawa (2007)'s main result.
- Example: mgf of exponential distribution with exponent α is
$$M(z) = \int_0^\infty \alpha \mathrm{e}^{-\alpha x} \mathrm{e}^{zx} \,\mathrm{d}x = \frac{\alpha}{\alpha - z},$$
so we can take $A = \alpha$ and $B = \infty$.

Proof of main result for iid case

- Let $\{X_t\}_{t=1}^{\infty}$ be iid with mgf $M_X(z) = \mathrm{E}[\mathrm{e}^{zX}]$.
- mgf of $W_T = \sum_{t=1}^{T} X_t$ when T is geometric with mean $1/p$ is

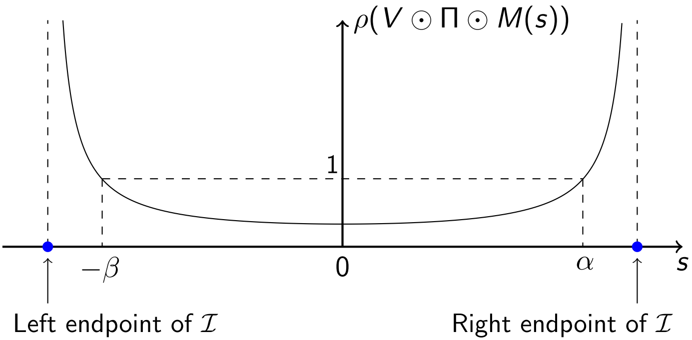
$$M_W(z) = \sum_{k=1}^{\infty} (1-p)^{k-1} p (M_X(z))^k = \frac{pM_X(z)}{1-(1-p)M_X(z)}.$$

- Since $M_X(z)$ holomorphic, pole of $M_W(z)$ satisfies $M_X(z) = \frac{1}{1-p}$.
- Using convexity of $M_X(s+it)$ with respect to s, easy to show pole is simple.
- Hence assumption of Tauberian theorem satisfied. Tail exponents satisfy

$$\mathrm{E}[\mathrm{e}^{\alpha X}] = \mathrm{E}[\mathrm{e}^{-\beta X}] = \frac{1}{1-p}.$$

▸ Skip general case

Strategy for general case



First show that $\rho(V \odot \Pi \odot M(s))$ is a convex function of s on $\mathcal{I}$...

Step 1: $\rho(V \odot \Pi \odot M(s))$ is convex

- In general, if $0 \leq A \leq B$ and $\theta \in (0,1)$, it is known that

$$\rho(A) \leq \rho(B) \quad \text{and} \quad \rho(A^{(\theta)} \odot B^{(1-\theta)}) \leq \rho(A)^\theta \rho(B)^{1-\theta}$$

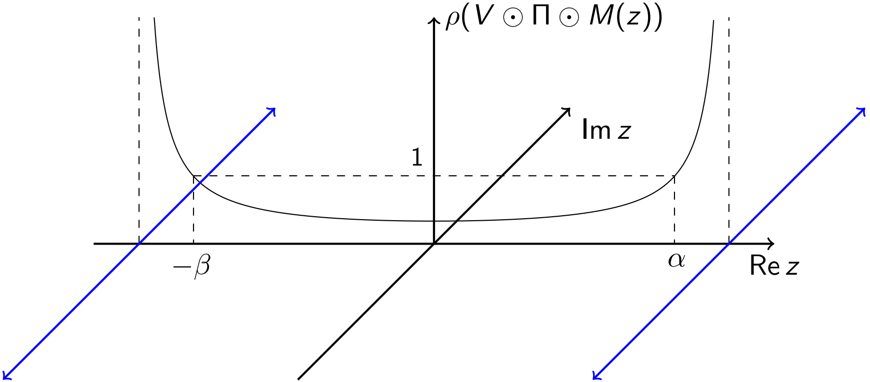
(element-wise power and product)

- Since M is a matrix of mgfs (which are log-convex), we have $M(\theta s_1 + (1-\theta)s_2) \leq M(s_1)^{(\theta)} \odot M(s_2)^{(1-\theta)}$. Hence

$$\begin{aligned}
&\rho(V \odot \Pi \odot M(\theta s_1 + (1-\theta)s_2)) \\
&\leq \rho(V \odot \Pi \odot M(s_1)^{(\theta)} \odot M(s_2)^{(1-\theta)}) \\
&= \rho((V \odot \Pi \odot M(s_1))^{(\theta)} \odot (V \odot \Pi \odot M(s_2))^{(1-\theta)}) \\
&\leq \rho(V \odot \Pi \odot M(s_1))^\theta \rho(V \odot \Pi \odot M(s_2))^{1-\theta},
\end{aligned}$$

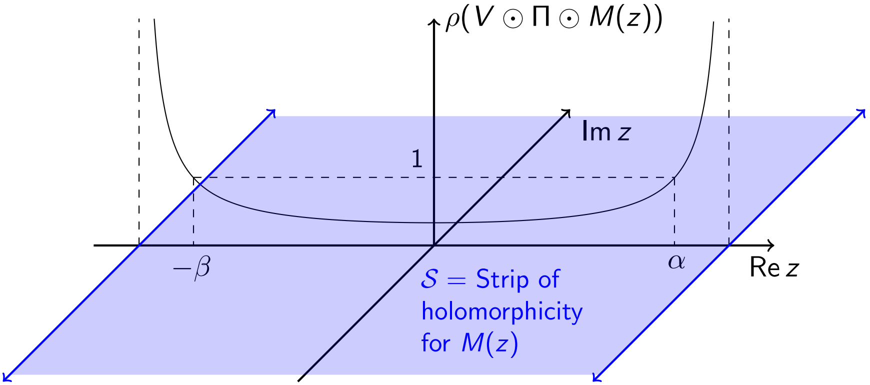
so $\rho(V \odot \Pi \odot M(s))$ is log-convex (hence convex).

Strategy for general case



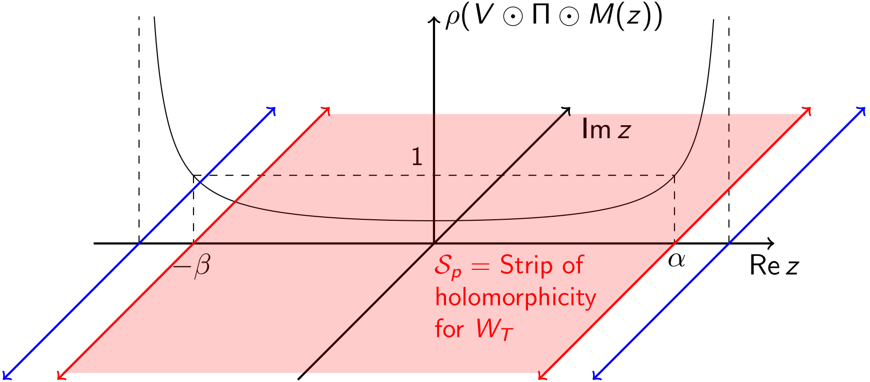
. . . then extend things to the complex plane . . .

Strategy for general case



. . . and observe that $M(z)$ is holomorphic on the strip $\mathcal{S} = \{z \in \mathbb{C} : \operatorname{Re} z \in \mathcal{I}\}$.

Strategy for general case



Finally, show that the Laplace transform of W_T is holomorphic on the strip $\mathcal{S}_p = \{z \in \mathbb{C} : -\beta < \operatorname{Re} z < \alpha\}$, check that it has poles at $-\beta$ and α, and apply Nakagawa's theorem.

Step 2: MGF of W_T

- Let $M_{W,n}(t,z) = \mathrm{E}\left[\mathrm{e}^{zW_{T\wedge t}} \,\middle|\, J_0 = n\right]$ be MGF of $W_{T\wedge t}$ conditional on $J_0 = n$.
- Using dynamic programming, can show

$$M_{W,n}(t,z) = \sum_{n'=1}^{N} \pi_{nn'} M_{nn'}(z)(1 - v_{nn'} + v_{nn'} M_{W,n'}(t-1,z)).$$

- Putting into a matrix and iterating, vector of MGF of W_T is

$$M_W(\infty, z) = A(z)^{-1}((E - V) \odot \Pi \odot M(z))e,$$

where $E = 1_{N\times N}$, $e = 1_{N\times 1}$, and $A(z) = I - V \odot \Pi \odot M(z)$. (Defined on $\mathcal{S}_p = \{z \in \mathbb{C} : -\beta < \operatorname{Re} z < \alpha\}$)

Remaining step: check α is a simple pole

- Since $\rho(V \odot \Pi \odot M(\alpha)) = 1$ and $V \odot \Pi \odot M(\alpha)$ is a nonnegative, irreducible matrix, by Perron-Frobenius theorem 1 is an eigenvalue of $V \odot \Pi \odot M(\alpha)$.
- Hence $A(\alpha) = I - V \odot \Pi \odot M(\alpha)$ is noninvertible.
- $A(z) = I - V \odot \Pi \odot M(z)$ is invertible in a neighborhood of α except at α ($\because$ $\det A(z)$ holomorphic on $\mathcal{S}$ and nonconstant, so zeroes are isolated).
- Since Perron root is simple, α is a simple pole of $A(z)^{-1}$, so we can apply Tauberian theorem.
- (Many more details to fill in, but refer to paper.)

Application 1: Wealth distribution

- We build an Aiyagari (1994)-type heterogeneous-agent model (but with investment risk) in a Markovian setting and show that wealth distribution has a Pareto tail.
- Two types of agents, capitalists and workers.
- For simplicity, workers get wage and consume everything (hand-to-mouth).
- Capitalists own capital, hire labor, and produce.
- Idiosyncratic investment risk. States: $s \in \mathcal{S} = \{1, \dots, S\}$, with transition probability matrix $P = (p_{ss'})$.

Capitalists

- Log utility (for simplicity; EZ in paper)

$$\mathrm{E}_0 \sum_{t=0}^{\infty} [\beta(1-p)]^t \log c_t,$$

where $\beta > 0$: discount factor, $p \in (0,1)$: bankruptcy probability.

- Constant returns to scale production function $F_s(k,l)$ in state s, say Cobb-Douglas $A_s k^\alpha l^{1-\alpha}$.
- Budget constraint:

$$c_t + k_{t+1} + \omega l_t = F_s(k_t, l_t) + (1-\delta)k_t,$$

where $\omega > 0$: wage, $\delta \in [0,1]$: depreciation rate.

- After bankruptcy, fraction $\kappa \in (0,1)$ of capital recycled back to economy as endowment to new capitalists.

Individual decision and equilibrium

Proposition

Given the wage $\omega > 0$, capital k, and state s, after choosing the optimal labor demand and consumption, the law of motion for capital becomes

$$k_{t+1} = \tilde{\beta} R_{s_t}(\omega) k_t,$$

where $\tilde{\beta} = \beta(1-p)$ is the effective discount factor and

$$R_s(\omega) = \alpha(1-\alpha)^{1/\alpha - 1} A_s^{1/\alpha} \omega^{1-1/\alpha} + 1 - \delta$$

is the gross return on capital.

Theorem

There exists a unique stationary equilibrium, and the wealth distribution has a Pareto upper tail with exponent $\zeta > 1$.

Numerical example

- Numerical example: $\beta = 0.96$, $p = 0.025$, capital share $\alpha = 0.38$, depreciation $\delta = 0.08$, capital recovery rate $\kappa = 0.8$.
- Log productivity is AR(1) with mean zero,

$$\log A_t = \rho \log A_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2),$$

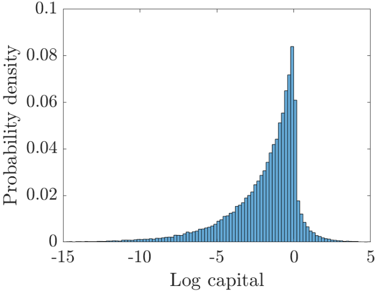
  where $\rho = 0.9$ and $\sigma = 0.1$, which we discretize using Farmer-Toda (2017 QE) method ($S = 9$ points).
- Pareto exponent $z = \zeta$ solves

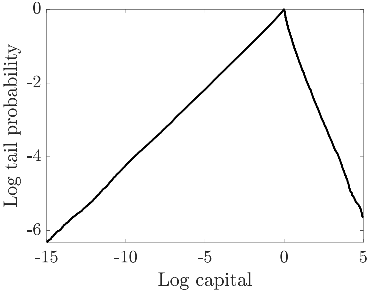
$$\lambda(z) := \rho(P \operatorname{diag}(\mathrm{e}^{z \log \tilde{\beta} R_s(\omega)})) = \frac{1}{1-p},$$

  and obtain $\zeta = 1.0814$.

Simulation

- Simulate economy with 100,000 agents and estimate tail exponent by maximum likelihood. $\widehat{\zeta} = 1.0485$, which is close to theoretical value.



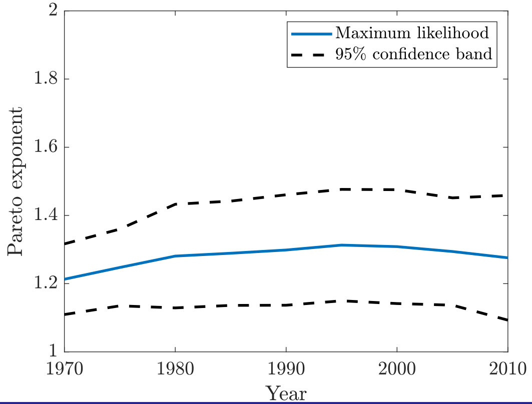


Application 2: Power law in Japanese municipalities

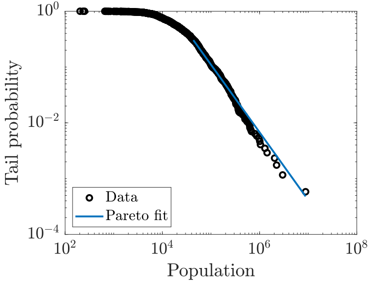
- Main question: are time series properties of population dynamics estimated from panel consistent with a stationary Pareto distribution estimated from cross-section?
- Data:
    - Japanese census data of municipality populations (1970–2010, every five years)
    - 1741 municipalities, but merge 23 Tokyo wards into "Tokyo city" (hence cross-sectional sample size $1741 - 23 + 1 = 1719$)
    - Spreadsheet available from Japanese Cabinet Office

Cross-sectional estimation

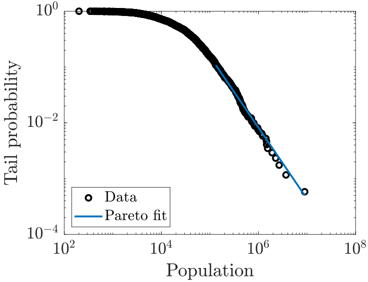
- Estimate Pareto exponent by maximum likelihood (Hill estimator).



Cross-sectional estimation



(a) 1970.



(b) 2010.

Panel estimation

- Assume relative size S_{it} of municipality i in year t follows random growth process
$$S_{i,t+1} = G_{i,t+1} S_{it},$$
where $G_{i,t+1}$: gross growth rate between year t and $t+1$.
- N-state Markov switching model with conditionally Gaussian shocks:
$$\log G_{i,t+1} \mid n_{it} = n \sim N(\mu_n, \sigma_n^2),$$
where state n_{it} evolves as a Markov chain with transition probability matrix Π.

Panel estimation

- Consider $N = 1, \dots, 5$; estimate parameters by maximum likelihood using Hamilton (1989) filter and expectation-maximization algorithm.
- Compute implied Pareto exponent by solving

$$(1-p)\rho(\Pi \operatorname{diag}(\mathrm{e}^{\mu_1 s+\sigma_1^2 s^2/2}, \dots, \mathrm{e}^{\mu_N s+\sigma_N^2 s^2/2})) = 1.$$

Panel estimation

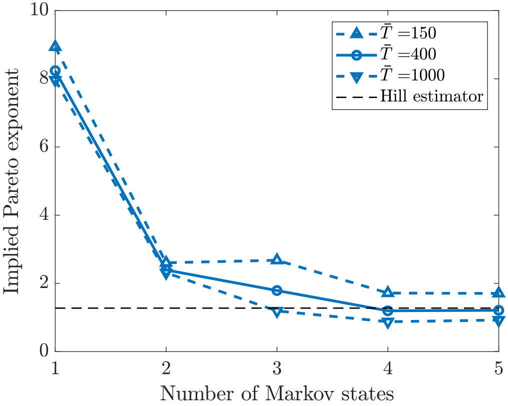
- Consider $N = 1, \dots, 5$; estimate parameters by maximum likelihood using Hamilton (1989) filter and expectation-maximization algorithm.
- Compute implied Pareto exponent by solving

$$(1-p)\rho(\Pi \operatorname{diag}(\mathrm{e}^{\mu_1 s+\sigma_1^2 s^2/2}, \dots, \mathrm{e}^{\mu_N s+\sigma_N^2 s^2/2})) = 1.$$

- Choosing mean age $\bar{T} = 1/p$:
  - Meiji Restoration is in 1868, so lower bound $\bar{T} = 150$.
  - Kamakura Shogunate started in 1185, so upper bound $\bar{T} = 1000$.
  - Tokugawa Shogunate started and moved capital to Tokyo in 1603, so $\bar{T} = 400$ reasonable.
  - Hence consider $p = 1/1000, 1/400, 1/150$.

Implied Pareto exponent

- With $N = 1$ (iid), $\alpha \approx 8 \gg 1$.



Conclusion

- Geometrically stopped Markovian random growth processes have Pareto tails under weak assumptions.
- Wealth distributions in heterogeneous-agent models have Pareto tails with investment risk.
  - Known for special models, but here general formula.
- In Japanese municipality population data, time series properties of growth process and cross-sectional distribution of levels can only be reconciled with Markovian dynamics.
- Our theoretical results provide a technical tool to study such models.

Markov additive process

Definition (Markov additive process)

Let $\mathcal{N} = \{1, \dots, N\}$ be a finite set. A bivariate Markov process $(W, J) = \{W_t, J_t\}_{t=0}^{\infty}$ on the state space $\mathbb{R} \times \mathcal{N}$ is called a *Markov additive process* if $\mathrm{P}(W_0 = 0) = 1$ and the increments of W are governed by J in the sense that

$$\mathrm{E}\left[f(W_{t+s} - W_t)g(J_{t+s}) \mid \mathcal{F}_t\right] = \mathrm{E}\left[f(W_s - W_0)g(J_s) \mid J_0\right]$$

for all $s, t \in \{0\} \cup \mathbb{N}$ and all nonnegative measurable functions f and g.

Go back