An Impossibility Theorem for Wealth in Heterogeneous-agent Models with Limited Heterogeneity

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Three motivating facts

1. Income and wealth distributions obey power law

$$P(X > x) \sim x^{-\alpha},$$

where α: Pareto exponent (Pareto, 1897).

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2. Wealth has heavier tail than income: $\alpha^{\text{wealth}} < \alpha^{\text{income}}$
   - $\alpha^{\text{wealth}} \approx 1.5$
     (Pareto, 1897; Klass *et al.*, 2006; Vermeulen, 2018)
   - $\alpha^{\text{income}} > 2$
     (Atkinson, 2003; Nirei & Souma, 2007; Toda, 2012)

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      (Atkinson, 2003; Nirei & Souma, 2007; Toda, 2012)
3. "Canonical" heterogeneous-agent macro models have difficulty explaining this
   (Aiyagari, 1994; Huggett, 1996; Castañeda *et al.*, 2003)

This paper

- We prove:

Theorem

In any "canonical" Bewley–Huggett–Aiyagari model, tail behavior of income and wealth are the same ($\alpha^{\text{wealth}} = \alpha^{\text{income}}$).

- "Canonical" means
  1. infinitely-lived agents,
  2. risk-free savings,
  3. constant discount factor
- These conditions are tight: relaxing any one of these assumptions can generate Pareto-tailed wealth distributions

Literature

Bounded income $\implies$ bounded wealth Schechtman & Escudero (1977), Aiyagari (1993), Huggett (1993), Açıkgöz (2018)

Impossibility result Benhabib, Bisin, & Luo (2017)

Possibility results

- Investment risk:
  Nirei & Souma (2007), Benhabib, Bisin, & Zhu (2011, 2015, 2016), Toda (2014)
- Random discount factor:
  Krusell & Smith (1998), Toda (2018)

Income fluctuation problem Chamberlain & Wilson (2000), Li & Stachurski (2014)

Light/heavy tail, exponential decay rate

- X: random variable; moment generating function: $M_X(s) = \mathrm{E}[\mathrm{e}^{sX}] \in [0, \infty]$
- We say X is *light-tailed* if $M_X(s) < \infty$ for some $s > 0$; otherwise *heavy-tailed*
- Since $M_X(s)$ convex, $\lambda = \sup \{s \geq 0 \mid M_X(s) < \infty\}$ well-defined
- If $s \in [0, \lambda)$, by Markov's inequality $P(X > x) \leq M_X(s)\mathrm{e}^{-sx}$
- Take log, divide by x, let $x \to \infty$, and $s \uparrow \lambda$; then

$$\limsup_{x \to \infty} \frac{\log P(X > x)}{x} = -\lambda$$

- We call λ *exponential decay rate* of X

Polynomial decay rate

- Since log of Pareto is exponential, if X heavy-tailed, natural to consider $\log X_+$, where $X_+ = X1_{X>0}$
- $M_{\log X_+}(s) = \mathrm{E}[\mathrm{e}^{s \log X_+}] = \mathrm{E}[X_+^s]$
- Define $\alpha = \sup \left\{ s \geq 0 \,\middle|\, \mathrm{E}[X_+^s] < \infty \right\}$
- Similarly, we can show

$$\limsup_{x \to \infty} \frac{\log P(X > x)}{\log x} = -\alpha,$$

polynomial decay rate

- Straightforward to define (uniform) decay rates for class of random variables $\{X_t\}_{t \in \mathcal{T}}$

Tail behavior of “contractive” processes

Theorem

Let $X_0 \geq 0$ be some real number and $\{X_t, Y_t\}_{t=1}^{\infty}$ be a nonnegative stochastic process such that

$$X_t \leq \rho X_{t-1} + Y_t$$

for all $t \geq 1$, where $0 \leq \rho < 1$. Then

1. *If $\{Y_t\}_{t=1}^{\infty}$ has a compact support, then so does $\{X_t\}_{t=1}^{\infty}$.*
2. *If $\{Y_t\}_{t=1}^{\infty}$ is uniformly light-tailed with exponential decay rate λ, then $\{X_t\}_{t=1}^{\infty}$ is uniformly light-tailed with exponential decay rate $\lambda' \geq (1-\rho)\lambda$.*
3. *If $\sup_t \mathrm{E}[Y_t] < \infty$ and $\{Y_t\}_{t=1}^{\infty}$ is uniformly heavy-tailed with polynomial decay rate α, then $\{X_t\}_{t=1}^{\infty}$ has a polynomial decay rate $\alpha' \geq \alpha$.*

Proof

- If $\{Y_t\} \subset [0, Y]$, then by iteration

$$\begin{aligned} X_t &\le Y_t + \rho Y_{t-1} + \cdots + \rho^{t-1} Y_1 + \rho^t X_0 \\ &\le (1 + \rho + \cdots + \rho^{t-1}) Y + \rho^t X_0 \\ &= \frac{1-\rho^t}{1-\rho} Y + \rho^t X_0 \le \frac{1}{1-\rho} Y + X_0 \end{aligned}$$

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- If $\{Y_t\}$ uniformly light-tailed, use Hölder
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- If $\{Y_t\}$ uniformly light-tailed, use Hölder
- If $\sup_t \mathrm{E}[Y_t] < \infty$, use Minkowski
- Same result holds if $X_t \le \phi(X_{t-1}) + Y_t$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that (i) ϕ is bounded on any bounded set, and (ii) $\rho := \limsup_{x \to \infty} \phi(x)/x < 1$

Income fluctuation problem

- In Bewley–Huggett–Aiyagari models, agents solve *income fluctuation problem*

$$\begin{aligned} &\text{maximize} && \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\text{subject to} && a_{t+1} = R(a_t - c_t) + y_{t+1}, \\ & && 0 \le c_t \le a_t \end{aligned}$$

- Here a_t: asset, c_t: consumption, y_t: income, $\beta > 0$: discount factor, $R > 0$: gross risk-free rate
- $c_t \le a_t$ implies no borrowing (wlog)

Existence of solution

Assumption

A1 *Utility function is twice continuously differentiable on* $\mathbb{R}_{++}$ *and satisfies* $u' > 0$, $u'' < 0$, $u'(0) = \infty$, *and* $u'(\infty) = 0$

A2 *Income process* $\{y_t\}$ *takes the form* $y_t = y(z_t)$, *where* $\{z_t\}$ *is a Markov process on some set* Z *and* $\sup_{z \in Z} \mathrm{E}\left[y(z_{t+1}) \mid z_t = z\right] < \infty$

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Proposition (Essentially Li & Stachurski (2014))

Suppose A1–A2 hold and $\beta R < 1$. *Then there exists a unique consumption policy function* $c(a, z)$ *that solves the income fluctuation problem. Furthermore, we have* $0 < c(a, z) \leq a$, c *is increasing in* a, *and* $c(a, z)$ *can be computed by policy function iteration.*

Policy function iteration

- If $c_t < a_t$, then Euler equation: $u'(c_t) = \mathrm{E}\left[\beta R u'(c_{t+1}) \mid z_t\right]$

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- In either case, $u'(c_t) = \max\left\{\beta R \,\mathrm{E}\left[u'(c_{t+1}) \mid z_t\right], u'(a_t)\right\}$

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- In either case, $u'(c_t) = \max\left\{\beta R\,\mathrm{E}\left[u'(c_{t+1}) \mid z_t\right], u'(a_t)\right\}$
- Let $\mathcal{C}$ be set of candidate consumption policy $c(a, z)$, define policy function operator $K : \mathcal{C} \to \mathcal{C}$ by $(Kc)(a, z) = t$, where

$$u'(t) = \max\left\{\beta R\,\mathrm{E}\left[u'(c(R(a-t)+y', z')) \mid z\right], u'(a)\right\}$$

- Can prove properties of $c(a, z)$ using convergence result in previous proposition

Linear lower bound on consumption

- To bound wealth from above, sufficient to bound consumption from below because $a' = R(a - c) + y'$
- With bounded relative risk aversion (BRRA), can obtain *linear lower bound* on consumption

A3 u is BRRA: $\bar{\gamma} = \sup_x -xu''(x)/u'(x) < \infty$

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Proposition

Suppose A1–A3 hold and $1 \le R < 1/\beta$. *Then for all* $m \in (1 - 1/R, 1 - \beta^{1/\bar{\gamma}} R^{1/\bar{\gamma}-1})$, *we have* $c(a, z) \ge ma$.

- Intuition: with impatience ($\beta R < 1$), agent consumes more than Permanent Income Hypothesis $c(a, z) = (1 - 1/R)a$

Skip proof

Step 1: $c(a,z) \geq c_0(a)$ (consumption with zero income)

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- If $t < c_0(a)$, then

$$
\begin{aligned}
u'(t) &> u'(c_0(a)) \\
&= \max \left\{ \beta R \operatorname{E} \left[u'(c_0(R(a - c_0(a)))) \mid z \right], u'(a) \right\} \\
&\geq \max \left\{ \beta R \operatorname{E} \left[u'(c_0(R(a - t) + y')) \mid z \right], u'(a) \right\} = u'(t),
\end{aligned}
$$

contradiction

Step 2: Implication of BRRA

Lemma

If u is BRRA, then for any $\kappa \in (0,1)$, we have $\inf_x (u')^{-1}(\kappa u'(x))/x > 1$*.*

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Lemma

If u is BRRA, then for any $\kappa \in (0,1)$, we have $\inf_x (u')^{-1}(\kappa u'(x))/x > 1$.

- Let $y = (u')^{-1}(\kappa u'(x))$
- Then for $\gamma(x) = -xu''(x)/u'(x)$,

$$-\log \kappa = \log u'(x) - \log u'(y) = -\int_1^{y/x} \frac{\partial}{\partial s} \log u'(xs) \, \mathrm{d}s$$
$$= -\int_1^{y/x} \frac{xu''(xs)}{u'(xs)} \, \mathrm{d}s = \int_1^{y/x} \frac{\gamma(xs)}{s} \, \mathrm{d}s \leq \bar{\gamma} \log \frac{y}{x}$$
$$\Longrightarrow \frac{y}{x} \geq \kappa^{-1/\bar{\gamma}} > 1$$

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- For $\bar{m} = 1 - \beta^{1/\bar{\gamma}} R^{1/\bar{\gamma}-1} \in (1 - 1/R, 1)$, can show

$$(\forall m \in (1 - 1/R, \bar{m}))(\forall a \geq 0)(t = (K_0 c)(a) \geq ma)$$

(This is most difficult part, which uses previous lemma)

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- Then $c(a) \leq (K_0^n c)(a) \to c_0(a) \leq c(a, z)$
- Hence $c(a, z) \geq c(a) = ma$

Impatience $\implies$ income and wealth same tail behavior

Proposition

Suppose A1–A3 hold and $\beta R < 1$. Let $\{a_t\}$ be the wealth arising from the solution to the income fluctuation problem. Then

1. *If $\{y_t\}$ is uniformly light-tailed, then so is $\{a_t\}$.*
2. *If $\{y_t\}$ is uniformly heavy-tailed with polynomial decay rate α, then $\{a_t\}$ has polynomial decay rate $\alpha' \geq \alpha$.*

Proof

- It suffices to show $a_{t+1} \leq \rho a_t + y_{t+1}$ for some $\rho \in [0, 1)$

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$$a_{t+1} = R(a_t - c_t) + y_{t+1} \leq \rho a_t + y_{t+1}$$

for $\rho = R < 1$

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  $$a_{t+1} = R(a_t - c_t) + y_{t+1} \leq \rho a_t + y_{t+1}$$
  for $\rho = R < 1$
- If $R \geq 1$, since $c(a,z) \geq ma$ for $m \in (1 - 1/R, 1 - \beta^{1/\bar{\gamma}} R^{1/\bar{\gamma}-1})$, we have
  $$a_{t+1} \leq R(1-m)a_t + y_{t+1} \leq \rho a_t + y_{t+1}$$
  for $\rho \in ((\beta R)^{1/\bar{\gamma}}, 1)$

Impossibility Theorem

Definition

A *Bewley–Huggett–Aiyagari model* is any dynamic general equilibrium model such that ex ante identical, infinitely-lived agents solve an income fluctuation problem.

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Theorem (Impossibility of heavy/heavier tails)

Consider a Bewley–Huggett–Aiyagari model such that A1–A3 hold. Suppose that an equilibrium with a wealth distribution with a finite mean exists and let $R > 0$ be the equilibrium gross risk-free rate. Then

1. *If income light-tailed, so is wealth.*
2. *If income heavy-tailed with polynomial decay rate α, then wealth has a polynomial decay rate $\alpha' \geq \alpha$.*

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- Hence if $\beta R > 1$, we have $u'(c_t) \to 0$ and $c_t \to \infty$, violating market clearing

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- Hence if $\beta R > 1$, we have $u'(c_t) \to 0$ and $c_t \to \infty$, violating market clearing
- Thus $\beta R \leq 1$ in equilibrium; theorem follows from previous result Skip possibility

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  - There is idiosyncratic investment risk, but risky investment limited to three values $\{k_1, k_2, k_3\}$

    $\Longrightarrow$ Reduces to case with additive income only

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- Cagetti & De Nardi (2006) is light-tailed
  - CRRA utility
  - There is idiosyncratic investment risk, but decreasing returns to scale $(\nu < 1)$:

$$a' = \theta k^\nu + (1 - \delta)k + (1 + r)(a - k) - c$$

$\implies$ Reduces to case with additive income only

Possibility results

- We have impossibility when
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- Can we get $\alpha^{\text{wealth}} < \alpha^{\text{income}}$ by relaxing these conditions?

Possibility results

- We have impossibility when
  1. infinitely-lived agents,
  2. risk-free savings, and
  3. constant discount factor
- Can we get $\alpha^{\text{wealth}} < \alpha^{\text{income}}$ by relaxing these conditions? Yes!
  1. OLG: Wold & Whittle (1957) (mechanical), Carroll *et al.*(2017), McKay (2017) (numerical)
  2. Idiosyncratic investment risk: Nirei & Souma (2007), Benhabib, Bisin, & Zhu (2011, 2015, 2016), Toda (2014), Toda & Walsh (2015), etc. (all analytical)
  3. Random discount factor: Krusell & Smith (1998) (numerical), Toda (2019) (analytical)
- Hence remaining case is OLG with analytical results

Model

- Finitely many types of agents $j = 1, \ldots, J$; $\pi_j \in (0,1)$: fraction of type j; $y_j > 0$: (constant) endowment
- Preferences are CRRA,

$$\mathrm{E}_0 \sum_{t=0}^{\infty} [\beta_j (1 - p_j)]^t \frac{c_t^{1-\gamma_j}}{1 - \gamma_j},$$

  where p_j: birth/death probability
- Agents trade only risk-free asset; R: gross risk-free rate
- $\tilde{R}_j = \frac{R}{1-p_j}$: effective risk-free rate faced by type j
- Consider stationary equilibrium

Wealth distribution is Pareto

- Budget constraint essentially $w' = \tilde{R}_j(w - c)$
- Optimal consumption rule $c = \left(1 - \tilde{\beta}_j^{1/\gamma_j}\tilde{R}_j^{1/\gamma_j - 1}\right) w$ as in Samuelson (1969), where $\tilde{\beta}_j = \beta_j(1 - p_j)$

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Theorem

A stationary equilibrium exists. ▸ Details *Furthermore,*

1. *If $\{\beta_j\}_{j=1}^J$ take at least two distinct values, then $\beta_j R > 1$ for at least one j and the stationary wealth distribution has a Pareto upper tail with exponent*

$$\alpha = \min_{j:\beta_j R > 1} \left[-\gamma_j \frac{\log(1 - p_j)}{\log(\beta_j R)} \right] > 1.$$

2. *If $\beta_1 = \cdots = \beta_J = \beta$, then $R = 1/\beta$ and the wealth distribution of each type is degenerate.*

Conclusion

- In canonical Bewley–Huggett–Aiyagari models with
  1. infinitely-lived agents,
  2. risk-free savings,
  3. constant discount factor,

  tail behavior of income and wealth are the same
- It was a 'folk theorem'; we have a formal proof
- To explain wealth distribution, need to relax at least one assumption; any will do (in paper)
- Which mechanism (birth/death, idiosyncratic investment risk, random discount factor) is most important is an empirical question

- Let W_j be aggregate wealth of type j
- By accounting, $W_j = (1-p_j)(\beta_j R)^{1/\gamma_j} W_j + p_j w_{j0}$, where

$$w_{j0} = \sum_{t=0}^{\infty} \tilde{R}_j^{-t} y_j = \frac{\tilde{R}_j}{\tilde{R}_j - 1} y_j$$

  is initial wealth of type j agent
- Hence $W_j = \frac{p_j w_{j0}}{1-(1-p_j)(\beta_j R)^{1/\gamma_j}}$
- Market clearing condition is

$$0 = \sum_{j=1}^{J} \pi_j (W_j - w_{j0}) = \sum_{j=1}^{J} \frac{R \pi_j y_j \left((\beta_j R)^{1/\gamma_j} - 1\right)}{\left(\frac{R}{1-p_j} - 1\right)\left(1 - (1-p_j)(\beta_j R)^{1/\gamma_j}\right)}$$